Why are Firms Different? The Role of Strategic Resources[1]

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Abstract

We analyze a continuous time model in which two firms make stochastic investments in stocks of complementary strategic resources, productive assets that take a long time to build and often can be used in more than one market. The option to diversify means that investments in strategic resources typically do not face saturating incentives – unlike what is the case with investments aimed at increasing share in a single market. In fact, because the strategic resources are complements, the firm with more of them will have stronger incentives to invest and will thus have a better than 50/50 chance of pulling further ahead. The result does not depend on whether larger investments pay off faster or in bigger pieces.

I. INTRODUCTION

It is a well-documented fact that the behavior and performance of firms differ, and that this matters a great deal. (See e. g. Gort (1963), Mueller (1986), Bloom and Van Reenen (2007), Hsie and Klenow (2009), Syverson (2011), Atalay, Hortacsu, and Syverson (2014), Bernard, Dhyne, Magerman, Manova, and Moxnes (2022), and many others.) The implications have been explored in several subfields of economics, including trade, macro, industrial organization, and

[1] The paper benefitted from comments by participants in MIT's Organizational Economics Lunch as well as the 2022 ISMS and SIOE conferences.

labor. In the management literature, the existence of inter-firm differences is the fundamental premise on which the resource-based view of the firm is built. As a result, research in many branches of management, including strategy, marketing, human resources, and operations, explores implications of differences between firms.

In spite of all this evidence and influence, we still do not understand why these differences exist. Specifically, we do not have generally accepted answers to two natural questions: (i) How did the differences appear in the first place? and (ii) why have they not been eliminated by imitation? We will here propose new answers to these questions. The answers are based on firms' attempts to gain competitive advantage through time-consuming investments in productive assets. Specifically, we look at a continuous time model in which two firms invest in a sequence of complementary productive assets and each investment succeeds at a random time. We find an equilibrium in which firms initially make identical decisions. However, once one of them has had better luck than its competitor, it faces larger returns to further investment (because of the complementarity). This makes it more likely that a leader will move further ahead as opposed to be caught by the laggard and as time passes, the firms become more and more likely to diverge.

We also make a more provincial contribution to model building technology. Standard results allow us to analyze continuous time models with Poisson dynamics if the sizes of individual jumps are subject to control while intensities are not. The more appealing case in which the arrival intensities, rather than the jump sizes, can be controlled, is not covered by standard theory because the dynamics do not satisfy a Lipschitz condition. In fact, this latter case has only been treated in two seemingly long-forgotten papers (Jacod and Protter, 1982 and Protter, 1983). We use the standard formulation in the main text but show in Appendix A that a natural version of the alternative gives essentially the same results. So the example is a partial justification for standard practice.

Literature

The analysis of competition with sequential investments has a long history in Industrial Organization going back to at least Arrow (1962). These are typically single industry models with the property that laggards eventually have higher marginal returns from investment and therefore will tend to catch up with leaders. (See Vickers, 1986, for an example and Reinganum, 1989, for a survey).[2] This "Arrow effect" is driven by the fact that the leader's upside incentives naturally dull as they get closer to dominance.

Some more recent papers in the IO tradition have identified specific competitive situations, often related to the possibility of the laggard exiting, under which the leader will stay ahead in perpetuity (Besanko and Doraszelski, 2004; Etro, 2004; and Doraszelski and Markovich, 2007).[3] Another tack is taken by Selove (2014) who looks at a market with two segments and two firms. If investments are segment specific, there is an equilibrium in which each firm monopolizes its own segment.

Other recent contributions have a more organizational flavor and again focus on specific examples, often looking at relational contracts or human resources (examples include Gibbons and Henderson, 2013; Powell, 2019; and Board, Meyer-ter-Vehn, and Sadzik, 2020). In many of these models the laggard is trapped in a less efficient equilibrium.

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[2] Interestingly, these results conflict with the extensive empirical evidence showing that larger firms spend more on R&D (Cohen and Levin, 1989).

[3] A firm that has exited is obviously exhibiting long term differences from one that has not.

The paper most closely related to ours is Athey and Schmutzler (2001). Like us, they look at enduring differences between firms making sequences of investments in productive assets. In their model, however, the differences appear because the firms' investments are strategic substitutes. We rely on the simpler assumption that a firm's own investments are complements, but combine it with the additional assumption that the productive assets can be deployed in more than one market.

Before justifying these two assumptions, we need to characterize the investments in more detail.

Strategic Resources

We focus on investments in "strategic resources" - productive assets that (1) take a long time to build and (2) are hard to trade in fractions.[4] These are the kinds of assets that confer a competitive advantage on the firm that owns or controls them.[5] Commonly mentioned examples of strategic resources include brand reputations, groups of loyal customers, IP, teams of employees with special skills, operational know-how, a corporate culture that promotes efficiency, and relationships with suppliers. Many, but not all,[6] of these assets have two additional properties: (3) They can be used in more than one industry, and (4) their capacity can be expanded at none or low costs. Some are partially non-rival: For example, the Gucci brand is

[4] The second property means that the assets have to be used inside the firm; you cannot make money by renting them out or selling off parts. The 1960's conglomerates failed when they tried to diversify around capital and management, two resources traded in very well-organized markets.
[5] Other commonly used terms, often with very similar meanings, are "organizational capital", "capabilities", "competencies", and simply "resources". We use the latter term because it suggests a broader range of assets and the modifier "strategic" is meant to exclude things like oil and steel.
[6] For example, the preemptive advantages Walmart obtained by being the first to locate in small towns does not have properties (3) and (4). However, many strategic resources do.

on many different products, eBay's many loyal customers allows it to sell several different products and services through its platform, Moderna's IP could be used to make many different vaccines, and good relationships with suppliers of a common input could help a firm compete in all industries that use the input in question. Others can relatively easily be "cloned" by slowly adding new employees: For example teams, know how, culture as in P&G. While this excess capacity cannot be sold, it allows the firms to quite cheaply gain competitive advantage in other markets - by diversifying. Specifically, the firm would be able to come into new markets with strategic resources for which it has paid relatively little while single business firms in those same markets would have to pay to develop them.

Key Claims.

We are now ready to defend the two key claims underlying our proposed explanation.

Claim A: Firms take future diversification options into account when they decide how much to invest in strategic resources.

Most large firms have diversified beyond their original markets: For example, if you want to understand what the New York Times and ABC are doing, you cannot assume that they only care about hard copy sales and prime time TV, respectively. America's five large tech companies provide another striking example. Alphabet, Amazon, Apple, Meta, and Microsoft all dominate their original market by a wide margin but instead of resting on their laurels, they are investing very, very large sums in new strategic resources: self-driving cars, space technology, virtual reality, quantum computing, chip design, robotics, etc..[7] Consistent with this a 2022 survey by Bain and Co. found that 58% of large acquisitions in 2021 were for "scope" rather than "scale", and Ding (2022) reports that 76% of all US manufacturing output was produced by multi-business firms. More systematic evidence and theory is offered by Benkard, Yurukoglu, and Zhang (2022), Boehm, Dhingra, and Morrow (2022), Chandler (1990), Chen, Elliott, and Koh (2022), Ding (2022), Ding, Fort, Redding, and Schott (2022), Wernerfelt (2022), and many others. We take a very brief look at the converse case in Appendix B.

Our second claim pertains to the sets of strategic resources assembled by firms:

Claim B: Firms often acquire strategic resources that complement those they already have.

One obvious reason for this is that complements are more valuable for the firm and confer a bigger advantage over competitors (Milgrom and Roberts, 1994). A more subtle reason is that the firm has an advantage in the "market" for these resources since it can pay more than rivals without complements. To see how, consider the process by which a firm with one strategic resource acquires a second. Just as a single strategic resource introduces an asymmetry on the output market, it creates another asymmetry in the competition for new strategic resources. This takes two forms: First, if the second strategic resource is complementary to the first, the firm can pay more for (compete harder to get) it. Second, the first strategic resource might make it cheaper to get the second (e. g. firms with efficient production processes can develop loyal customers at lower costs, etc.). In the present paper, we assume that the costs of developing such strategic resources are the same for all firms, but that complementarities cause their value to be

[7] For more detail see The Economist (January 22, 2022, pp. 16-18).

larger for firms with a bigger initial stock. For this claim we also take a look at the converse case in Appendix B.

II. MODEL

Two firms, X and Y, compete in continuous time. The outcome of market competition at time t depends on the firms' stocks of strategic resources, $(x_t, y_t) \epsilon R^{+2}$, $x_0 = y_0 = 0$. At time $T < \infty$, the market disappears, and all resources become worthless. Until then, the stocks of X and Y grow according to independent Poisson processes with intensity λ and jumps of size (u_t, v_t), respectively. To speak directly to the issue of catchup, we imagine that there is an ordered set of resources and that firms have to acquire them in order.[8] Once both firms have a resource, it confers a smaller competitive advantage and is thus less valuable. The stocks of x_t and y_t depreciate at rate φ as third parties develop substitutes.

The size of the jumps in the firms' stocks are chosen by the firms as C^2 Markov controls; firm X 's strategy is $u(x_t, y_t, t)$ and firm Y's is $v(y_t, x_t, t)$. The stocks of strategic resources therefore grow over time according to

$$(1) \quad x_t = -\int_0^t \boldsymbol{\varphi} x_s ds + \int_0^t \boldsymbol{u}(x_s, y_s, s) dN[\lambda]ds$$

$$(2) \quad y_t = -\int_0^t \boldsymbol{\varphi} y_s ds + \int_0^t \boldsymbol{v}(y_s, x_s, s) dM[\lambda]ds$$

where $N[\lambda]$ and $M[\lambda]$ are independent Poisson processes with intensity λ and the integrals sum the jumps at arrivals between 0 and t.[9]

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8 The model can also be thought of as representing a situation in which the firms can choose which resource to acquire next. In that case the model and the results still hold, but we have to interpret payoffs somewhat differently.

9 The problem formulation suggests a filtered probability space with time index, and we will conduct the analysis in that context.

Firms X and Y maximize $\int_0^T e^{-\rho t}[\gamma x_t^2 - \eta y_t^2 - ½u_t{}^2]dt$ and $\int_0^T e^{-\rho t}[\gamma y_t^2 - \eta x_t^2 - ½v_t^2]dt$, respectively, and we assume that $(\varphi, \rho, \eta) \epsilon R^{+3}$, and $\gamma > \eta$. This formulation implies that investments in individual strategic resources succeed at random (Poisson arrival) times and are subject to "time-compression diseconomies" in the sense that there are increasing costs to increasing the expected number of arrivals per unit of time (Dierickx and Cool, 1989). Finally, firms' quadratic returns to the stocks of resources reflect complementarities between them.[10]

Before analyzing this model, we want to acknowledge that the formulation is a bit unnatural: It would be preferable to assume that investments are focused on one strategic resource at a time but that firms can reduce the expected time to success by investing at higher rates. Analysis of such a formulation is, unfortunately, quite complicated since the dynamics become non-Lipschitzian. Perhaps as a result, the literature has paid no attention to it except for Jacod and Protter (1982), and Protter (1983). We nevertheless analyze a model with controlled arrival intensity in Appendix A. In a very natural analogue of the model presented here, we obtain a limiting result according to which the value - and policy functions are very similar to what we find here. This shows that our result does not depend on the unnatural formulation and suggests that the standard practice in the field is defensible.

Returning to the formulation with constant intensity, by Theorem 3.1 in Oeksendal and Sulem (2004), the firms want to find C^2 value functions $W(x_t, y_t, t)$, $W(y_t, x_t, t)$ from $R^{+3} \rightarrow R$ that satisfy the Bellman equations:[11]

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10 It would be preferable to replace the quadratic terms with powers of arbitrary degree above *1*, but I have not been able to analyze that case.

11 Since the value functions are C^2, we can use the infinitesimal generators of x_t and y_t to describe the movements of the value functions in infinitesimal time-intervals. The notation reflects that we will work with the right continuous with left limits (cadlag) versions of all processes, x_t, y_t, u_t, and v_t., such that the jump at t is $x_t - x_{t-}$ etc. The

$$(3)\ Max_{u(x,\,y,\,t)}\{e^{-\rho t}[\gamma x_t^2 - \eta y_t^2 - ½u_t^2] + \partial W(x_t, y_t, t)/\partial t - \varphi x_t \partial W(x_t, y_t, t)/\partial x - \varphi y_t \partial W(x_t, y_t, t)/\partial y + \lambda[W(x_{t-} + u_{t-}, y_{t-}, t_-) - W(x_{t-}, y_{t-}, t_-)] + \lambda[W(x_{t-}, y_{t-} + v^*_{t-}, t_-) - W(x_{t-}, y_{t-}, t_-)]\} = 0,$$

$W(x_T, y_T, T) = 0$, for firm X, and

$$(4)\ Max_{v(y,\,x,\,t)}\{e^{-\rho t}[\gamma y_t^2 - \eta x_t^2 - ½v_t^2] + \partial W(y_t, x_t, t)/\partial t - \varphi y_t \partial W(y_t, x_t, t)/\partial y - \varphi x_t \partial W(y_t, x_t, t)/\partial x + \lambda[W(y_{t-} + v_{t-}, x_{t-}, t_-) - W(y_{t-}, x_{t-}, t_-)] + \lambda[W(y_{t-}, x_{t--} + u^*_{t-}, t_-) - W(y_{t-}, x_{t-}, t_-)]\} = 0,$$

$W(y_T, x_T, T) = 0$, for firm Y.

The (Markov) equilibrium controls u^*_{t-} and v^*_{t-} are therefore given by the first order conditions

$$(5)\ u^*_t(x_t, y_t, t) = e^{\rho t}\lambda \partial W(x_t, y_t, t)/\partial x$$

$$(6)\ v^*_{t-}(y_t, x_t, t) = e^{\rho t}\lambda \partial W(y_t, x_t, t)/\partial y$$

Since the firms face symmetric problems, we will henceforth focus on firm X. Substitution of (5) and (6) into (3) allows us to rewrite the Bellman equation as:

$$(7)\ e^{-\rho t}(\gamma x_t^2 - \eta y_t^2) - e^{\rho t}\lambda^2[\partial W(x_t, y_t, t)/\partial x]^2/2 + \partial W(x_t, y_t, t)/\partial t - \varphi x_t \partial W(x_t, y_t, t)/\partial x - \varphi y_t \partial W(x_t, y_t, t)/\partial y + \lambda[W(x_{t-} + e^{\rho t}\lambda \partial W(x_t, y_t, t)/\partial x, y_{t-}, t_-) - W(x_{t-}, y_{t-}, t_-)] + \lambda[W(x_{t-}, y_{t-} + e^{\rho t}\lambda \partial W(y_t, x_t, t)/\partial y, t_-) - W(x_{t-}, y_{t-}, t_-)] = 0,\quad W(x_T, y_T, T) = 0$$

We guess a solution of the form

$$(8)\ W^g(x_t, y_t, t) \equiv e^{-\rho t}[a(t)x_t^2 + b(t)x_t + c(t) + d(t)y_t^2 + e(t)y_t],$$

$$a(T) = b(T) = c(T) = d(T) = e(T) = 0.$$

Using that the strategies are symmetric, this gives

$$(9)\quad u^g_{t-} = \lambda[2a(t)x_{t-} + b(t)]$$

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appearance of the – subscript in the terms for jumps in both x_t and y_t does not mean that they jump at the same time but reflects that *(7)* looks at expected values.

$$(10)\quad v^g_{t-} = \lambda[2a(t)y_{t-} + b(t)]$$

$$(11)\quad \lambda[W^g(x_{t-} + \lambda 2a(t)x_{t-} + \lambda b(t), y_{t-}, t_-) – W^g(x_{t-}, y_{t-}, t_-)] =$$

$$e^{-\rho t}\lambda\{4\lambda a(t)^2[a(t)\lambda + 1]x_{t-}^2 + 4\lambda a(t)b(t)[\lambda a(t) + 1]x_{t-} + \lambda b(t)^2[\lambda a(t) + 1]\}.$$

$$(12)\quad \lambda[W^g(x_{t-}, y_{t-}+ \lambda 2a(t)y_{t-} + \lambda b(t), t_-) – W^g(x_{t-}, y_{t-}, t_-)] =$$

$$e^{-\rho t}\lambda\{4\lambda a(t)d(t)[a(t)\lambda + 1]y_{t-}^2 + 2\lambda[2\lambda a(t)b(t)d(t) + a(t)e(t) +b(t)d(t)]y_{t-} +$$

$$\lambda b(t)[\lambda b(t)d(t) +e(t)]\}.$$

Rewriting the Bellman equation in terms of the coefficients in $W^g(x_{t-}, y_{t-}\ t)$ gives:

$$(13)\quad \gamma x_{t-}^2 – \eta y_{t-}^2 – 2\lambda^2 a(t)^2 x_{t-}^2 - 2\lambda^2 a(t)b(t)x_{t-} - \lambda^2 b(t)^2/2 + [a'(t) – \rho a(t)]x_t^2 + [b'(t) –$$

$$\rho b(t)]x_t +[c'(t) – \rho c(t)]+ [d'(t) – \rho d(t)]y_t^2+ [e'(t) – \rho e(t)]y_t\ - \varphi x_{t-}[2a(t)x_{t-} +b(t)] -$$

$$\varphi y_{t-}[2d(t)y_{t-} + e(t)] + 4\lambda^2 a(t)^2[a(t)\lambda + 1]x_{t-}^2 + 4\lambda^2 a(t)b(t)[\lambda a(t) + 1]x_{t-} + \lambda^2 b(t)^2[\lambda a(t)$$

$$+ 1] + 4\lambda^2 a(t)d(t)[a(t)\lambda + 1]y_{t-}^2 + 2\lambda^2[2\lambda a(t)b(t)d(t) + a(t)e(t) + b(t)d(t)]y_{t-} +$$

$$\lambda^2 b(t)[\lambda b(t)d(t) +e(t)] = 0,$$

$$a(T) = b(T) = c(T) = d(T)= e(T) = 0.$$

$W^g(x_{t-}, y_{t-}\ t)$ solves *(7)* if the constant term and the coefficients on x_t^2, x_t, y_t^2, and y_t all are zero at T. This means that the coefficients in $W^g(x_t, y_t, t)$ solve the following differential equations

$$(14)\quad a'(t) = - \gamma - 4\lambda^3 a(t)^3 + 2\lambda^2 a(t)^2 + [2\varphi + \rho]a(t),\ a(T) = 0$$

$$(15)\quad b'(t) = [- 4\lambda^3 a(t)^2 - 2\lambda^2 a(t) + \varphi + \rho]b(t),\ \ b(T) = 0$$

$$(16)\quad c'(t) = - \lambda^2 b(t)[\lambda a(t)b(t) + b(t)/2 + \lambda b(t)d(t) + e(t)] +\rho c(t),\ c(T) = 0$$

$$(17)\quad d'(t) = \eta\ + [– 4\lambda^2 a(t)\{\lambda a(t) + 1\} +2\varphi + \rho]d(t),\ d(T) = 0$$

$$(18)\quad e'(t) = - 2\lambda^2[2\lambda a(t)b(t)d(t) + a(t)e(t) + b(t)d(t)] +[\varphi + \rho]e(t),\ e(T) = 0$$

Observe first that $b(t) = e(t) = c(t) = 0$. Since (14) and (17) is a system of ODEs with continuous right-hand sides, a unique solution exists and we can conclude that:

Lemma 1: *$W^*(x_t, y_t, t) = e^{-\rho t}[a(t)x_t^2 + d(t)y_t^2]$ solves the HJB equation (8), and the Markov equilibrium strategies are $u^*_t = 2\lambda a(t)x_t$ and $v^*_t = 2\lambda a(t)y_t$.*

Lemma 2: *$a(t) > 0$ for all $t < T$.*

Proof: From *(14):* First, since $a'(t) < 0$ for t close to T, $a(t) > 0$ in that neighborhood. Second, since $a'(s) < 0$ if $a(s) = 0$, $a(s)$ cannot change sign and is therefore positive for all $t < T$.

QED

So the firm which, at a given t, has a greater stock of strategic resources will invest more and therefore be more likely to increase its stock. We thus get our main result:

Proposition 1: *$E_{t=0}|x_{s-} - y_{s-}|$ grows with s.*

Proof: Suppose that X is ahead at time h. The players are equally likely to get the next arrival at time $h + i$, but X will invest more, so $|x_h - y_h|$ will grow more if he gets the arrival than if Y does. The expected value of $|x_{h+i} - y_{h+i}|$ is therefore larger than $|x_h - y_h|$.The same mechanism applies for all later arrivals, and if Y gets the first arrival. So while $E_{t=0}(x_s - y_s) = 0$, $E_{t=0}|x_s - y_s|$ is strictly positive and grows with s.

QED

This means that firms, on the average, grow more different, leapfrog each other less frequently, and diversify more as they get older. At the industry level, we would expect an increasingly skewed size distribution and growing aggregate profits.

III. DISCUSSION

We have presented a model in which two firms invest in stocks of complementary strategic resources. The firms are initially identical and pursue the same strategy, but since their investments succeed at random times, their stocks will soon differ. Because the strategic resources are complements, the firm with more of them will have stronger incentives to invest and will thus have a better than 50/50 chance of pulling further ahead. The expected differences between the firms will grow over time as will the probability that they are different at all.

The critical assumptions are that investments are complements and that incentives do not saturate. The latter is a logical assumption when firms are investing in strategic resources in order to gain competitive advantages. These assets tend to have or develop excess capacity that can be used in several industries thus giving firms incentives to diversify and continue investing.

The current formulation, in which firms control the size of each success but not the intensity with which they arrive, is less appealing than one in which investments are focused on one strategic resource at a time but larger investments reduce the expected time to arrival. As will be apparent from the proof of *Observation 1,* it is significantly harder to obtain results with the latter formulation. However, the Observation also shows that the easier but less appealing formulation yield results that are very similar.

Beyond the application considered here, the Point Process techniques used in the main paper and Appendix A may be of independent interest as an alternative formulation of continuous time models with Brownian dynamics such as Ke and Villas-Boas (2019) and Ortner (2019) among many others.

APPENDIX A:

AN ALTERNATIVE FORMULATION WITH CONTROLLED INTENSITY

The model is the same as in the main text with two differences: First, the firms' resource stocks grow in unit sized jumps that arrive according to non-homogeneous Poisson processes with intensities (u_t, v_t), respectively. Second, to satisfy the sufficient condition in Jacod and Protter (1982) we need to bound the arrival intensities. However, since the bound does not appear in the original formulation and is irrelevant for the economic intuition, we will soon be taking it to infinity.

Formally, the intensities are chosen by the firms as C^2 Markov controls; firm X 's strategy is $u(x_t, y_t, t) \leq U$ and firm Y's is $v(y_t, x_t, t) \leq U$. The stocks of strategic resources therefore develop over time according to

$$(19) \quad x_t = -\int_0^t \boldsymbol{\varphi} x_s ds + \int_0^t \boldsymbol{dN}[\boldsymbol{u}(x_{s-}, y_{s-}, s-)]ds$$

$$(20) \quad y_t = -\int_0^t \boldsymbol{\varphi} y_s ds + \int_0^t \boldsymbol{dM}[\boldsymbol{v}(y_{s-}, x_{s-}, s-)]ds$$

where $N[u(x_s, y_{s-}, s-)]$ and $M[v(y_{s-}, x_{s-}, s-)]$ are Point processes with jumps of size 1 and intensities $u(x_{s-}, y_{s-}, s-)$ and $v(y_{s-}, x_{s-}, s-)$, respectively. Firms X and Y again maximize $\int_0^T \boldsymbol{e}^{-\rho t}[\gamma x_t^2 - \eta y_t^2 - ½u_t^2]dt$ and $\int_0^T \boldsymbol{e}^{-\rho t}[\gamma y_t^2 - \eta x_t^2 - ½v_t^2]dt$, respectively, and we assume that $(\varphi, \rho, \eta) \in R^{+3}$, and $\gamma > \eta$.

The problem no longer satisfies the conditions of Theorem 3.1 in Oeksendal and Sulem (2004) because jump intensities $u_{t-}(x_{t-}, y_{t-}, t-)$ and $v_{t-}(y_{t-}, x_{t-}, t-)$ depend on the very states they govern. Since the coefficients in *(19)* and *(20)* therefore do not satisfy Lipschitz conditions, we cannot apply classical existence and uniqueness results. Fortunately, we can rely on a more general result first obtained by Jacod and Protter (1982, Corollary 31) and Protter (1983, Corollary 3.11).[12] They show that *(19)* and *(20)* have solutions that are unique in law if there exists finite-valued increasing processes p and q such that $\int_0^t \boldsymbol{u}(x_s, y_s, s)ds \leq p_t$ and $\int_0^t \boldsymbol{v}(y_s, x_s, s)ds \leq q_t$ for all $t \geq 0$.[13] These conditions are clearly satisfied by $p_t = Ut$ and $q_t = Ut$ in our model.

Depending on U and the realizations of *(19)* and *(20)*, we can have up to four regions: (i) neither firm is constrained, (ii) one firm is constrained, (iii) the other firm is constrained, and (iv) both firms are constrained. We know that the firms start out and end unconstrained but in between they can visit all four regions several times and the solution would need to meet value matching and smooth pasting conditions (in three dimensions) on each occasion. The obvious difficulties associated with finding an explicit solution to this equation probably explains why the case with controlled intensities is looked at so rarely.[14]

In our case, we will settle for a limiting result:

Proposition 2: *Suppose that all jumps are of size 1 but that the arrival intensities* (u_t, v_t) *are controlled by firms X and Y, respectively. Suppose further that both intensities are bounded by U. In this case, as* $U \to \infty$

[12] The idea in the proof is to inductively define an increasing sequence of stopping times (jump times) $\tau_1, \tau_2, \dots \tau_n, ..$ and piece together the entire solution from the intervals $[\tau_1, \tau_2-)$ on which classical theory applies.
[13] Uniqueness in law means that all solutions produce the same distribution at all time steps given the same starting point and time. Classical conditions give path-wise uniqueness, a stronger property under which all solutions follow the same paths everywhere.
[14] The four pre-pasting value functions are all quadratic, so some progress may still be possible.

- *The Markov equilibrium strategies converge to*

$$u^*_{t-} = 2a(t)x_{t-} + a(t) + b(t), \text{ and } v^*_{t-} = 2a(t)y_{t-} + a(t) + b(t)$$

$$\text{where } a'(t) = -\gamma - 2a(t)^2 + 2\varphi a(t) + \rho a(t),\ a(T) = 0, \text{ and}$$

$$b'(t) = -[a(t) + b(t)]2a(t) + \varphi b(t) + \rho b(t),\ b(T) = 0,$$

In addition, the probability of the following converges to 1

- $Max_t\{u_{t-}, v_{t-}\} < U$,

- $a(t) > 0$, and

- $E_{t=0}|x_{s-} - y_{s-}|$ *grows with s.*

Proof: In the unconstrained problem, the Bellman equation for firm *X* is:

(21) $Max_{u(x,\,y,\,t)}\{e^{-\rho t}[\gamma x_t^2 - \eta y_t^2 - \frac{1}{2}u_t^2] + \partial W(x_{t-}, y_{t-}, t_-)/\partial t - \varphi x_t \partial W(x_t, y_t, t)/\partial x - \varphi y_t \partial W(x_t, y_t, t)/\partial y$
$+ u_{t-}[W(x_{t-} + 1, y_{t-}, t_-) - W(x_{t-}, y_{t-}, t_-)] + v^*_{t-}[W(x_{t-}, y_{t-}+1, t_-) - W(x_{t-}, y_{t-}, t_-)]\} = 0.$

The (Markov) equilibrium controls u^*_{t-} and v^*_{t-} are therefore given by the first order conditions

$$(22)\ u^*_{t-}(x_{t-}, y_{t-}, t_-) = e^{\rho t}[W(x_{t-} + 1, y_{t-}, t_-) - W(x_{t-}, y_{t-}, t_-)]$$

$$(23)\ v^*_{t-}(y_{t-}, x_{t-}, t_-) = e^{\rho t}[W(y_{t-} + 1, x_{t-}, t_-) - W(y_{t-}, x_{t-}, t_-)]$$

We can substitute (22) and (23) into (21) and rewrite the Bellman equation as:

$$(24)\ e^{-\rho t}(\gamma x_t^2 - \eta y_t^2) + \partial W(x_{t-}, y_{t-}, t_-)/\partial t - \varphi x_t \partial W(x_t, y_t, t)/\partial x - \varphi y_t \partial W(x_t, y_t, t)/\partial y + e^{\rho t}[W(x_{t-} +1, y_{t-}, t_-) - W(x_{t-}, y_{t-}, t_-)]^2/2 + e^{\rho t}[W(y_{t-} +1, x_{t-}, t_-) - W(y_{t-}, x_{t-}, t_-)][W(x_{t-}, y_{t-} +1, t_-) - W(x_{t-}, y_{t-}, t_-)] = 0.$$

We again guess a solution of the form

$$(25)\ W^g(x_t, y_t, t) \equiv e^{-\rho t}[a(t)x_t^2 + b(t)x_t + c(t) + d(t)y_t^2 + e(t)y_t].$$

Substituting (25) into (24) gives:

$$(26)\ [\gamma + 2a(t)^2 - 2\varphi a(t) + a'(t) - \rho a(t)]x_t^2 + [b'(t) - \rho b(t) - \varphi b(t) + 2a(t)\{a(t) + b(t)\}]x_t + [c'(t) - \rho c(t) + [a(t) + b(t)][d(t) + e(t)] + ½\{a(t) + b(t)\}^2] + [-\eta + d'(t) - \rho d(t) + 4a(t)d(t) - 2\varphi a(t)]y_t^2 + [e'(t) - \rho e(t) + 2d(t)b(t) + 4a(t)d(t) + 2a(t)e(t) - \varphi b(t)]y_t = 0$$

Our guess $W^g(x_{t-}, y_{t-} t_-)$ therefore solves *(26)* if the coefficients in $W^g(x_{t-}, y_{t-}, t_-)$, solve the following differential equations

$$(27)\ a'(t) = -\gamma - 2a(t)^2 + 2\varphi a(t) + \rho a(t),\ a(T) = 0$$

$$(28)\ b'(t) = -[a(t) + b(t)]2a(t) + \varphi b(t) + \rho b(t),\ b(T) = 0$$

$$(29)\ c'(t) = -[a(t) + b(t)]^2/2 - [a(t) + b(t)][d(t) + e(t)] + \rho c(t),\ c(T) = 0$$

$$(30)\ d'(t) = \eta - 4a(t)d(t) + 2\varphi a(t) + \rho d(t),\ d(T) = 0$$

$$(31)\ e'(t) = -2d(t)[a(t) + b(t)] - 2a(t)[d(t) + e(t)] + \varphi b(t) + \rho e(t),\ e(T) = 0.$$

As in the model with controlled jump sizes, we find that $a(t) \geq 0$ and that $u^g_{t-} = 2a(t)x_{t-} + a(t) + b(t)$. So in this region, it makes little difference whether it is the sizes or intensities of jumps that are controlled. However, since the Jacod-Protter results apply to the constrained model, we can only look at the limit in which U is so large that the probability of even hitting it is vanishing.

The complete solution to the constrained problem, call $W^*(x_{t-}, y_{t-}, t_-)$, differs from $W^g(x_{t-}, y_{t-}, t_-)$ because it has to take into account the possibility that the system may transition to other regions. In particular, the value function in region (i), has to reflect the possibility that constraint later may bind for one or both of the firms. The probability that this happens depends on U as well as the realizations of *(19)* and *(20)*. If a firm has a sequence of early arrivals and therefore a large stock of resources at an early date, it would want to invest at a high level and could hit the constraint. On the other hand, if U is large and the firm accumulates resources slowly, it is less likely to ever be constrained (because the incentives to invest become smaller as t approaches T). Since the probability of realizations where the constraints on u_{t-} and v_{t-} never bind goes to 1 as U goes to infinity, the result follows.

QED

Corollary 1: *In the model with quadratic costs of intensity and fixed jump size, as $U \rightarrow \infty$, the policy and value functions converge to the same polynomial order(linear and quadratic) as the corresponding functions in the model with fixed intensity and quadratic costs of jump size.*

Proof: By comparing the proofs of Proposition 2 and Lemma 1.

QED

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APPENDIX B: NECESSARY ASSUMPTIONS

Result 1 (Investments not Complements): *Suppose that the objectives are* $\int_0^T e^{-\rho t}[\gamma x_t - \eta y_t - \frac{1}{2}u_t{}^2]dt$ *and* $\int_0^T e^{-\rho t}[\gamma y_t - \eta x_t - \frac{1}{2}v_t^2]dt$*, such that the investments are neither complements nor substitutes. In that case,* $W^g(x_t, y_t, t)$ *changes to* $e^{-\rho t}[b(t)x_t + c(t) + e(t)y_t]$*,* $b(T) = c(T) = e(T) = 0$*; the Markov equilibrium strategies are* $u^*_{t-} = v^*_{t-} = b(t)$*; and the leader does not invest more than the laggard.*

Proof: The Bellman equation for X is

$$(37) Max_{u(x,\, y,\, t)}\{e^{-\rho t}[\gamma x_t - \eta y_t - \tfrac{1}{2}u_t^2] + \partial W(x_t, y_t, t)/\partial t - \varphi x_t \partial W(x_t, y_t, t)/\partial x - \varphi y_t \partial W(x_t, y_t, t)/\partial y + \lambda[W(x_{t-} + u_{t-}, y_{t-}, t_-) - W(x_{t-}, y_{t-}, t_-)] + \lambda[W(x_{t-}, y_{t-} + v^*_{t-}, t_-) - W(x_{t-}, y_{t-}, t_-)]\} = 0,\ W(x_T, y_T, T) = 0.$$

After substituting in the first order conditions we get

$$(38) e^{-\rho t}(\gamma x_t - \eta y_t) - e^{\rho t}\lambda^2[\partial W(x_{t-}, y_{t-}, t_-)/\partial x]^2/2 + \partial W(x_t, y_t, t)/\partial t - \varphi x_t \partial W(x_t, y_t, t)/\partial x - \varphi y_t \partial W(x_t, y_t, t)/\partial y + \lambda[W(x_{t-} + e^{\rho t}\lambda \partial W(x_{t-}, y_{t-}, t_-)/\partial x, y_{t-}, t_-) - W(x_{t-}, y_{t-}, t_-)] + \lambda[W(x_{t-}, y_{t-} + e^{\rho t}\lambda \partial W(y_{t-}, x_{t-}, t_-)/\partial y, t_-) - W(x_{t-}, y_{t-}, t_-)] = 0,\ W(x_T, y_T, T) = 0.$$

Guessing again $W^g(x_t, y_t, t) \equiv e^{-\rho t}[a(t)x_t^2 + b(t)x_t + c(t) + d(t)y_t^2 + e(t)y_t]$, $a(T) = b(T) = c(T) = d(T) = e(T) = 0$, allows us to rewrite the Bellman equation as:

$$(39) \gamma x_{t-} - \eta y_{t-} - \lambda^2 2a(t)^2 x_{t-}^2 - \lambda^2 2a(t)b(t)x_{t-} - \lambda^2 b(t)^2/2 + [a'(t) - \rho a(t)]x_t^2 + [b'(t) - \rho b(t)]x_t + [c'(t) - \rho c(t)] + [d'(t) - \rho d(t)]y_t^2 + [e'(t) - \rho e(t)]y_t - \varphi x_t[2a(t)x_t + b(t)] - \varphi y_t[2d(t)y_t + e(t)] + 4a(t)^2\lambda^2[a(t)\lambda + 1]x_{t-}^2 + 2\lambda^2 a(t)b(t)[2\lambda a(t) + 1]x_{t-} + \lambda^2 b(t)^2[\lambda a(t) + 1] + 4a(t)d(t)\lambda^2[a(t)\lambda + 1]y_{t-}^2 + 2\lambda^2 d(t)b(t)[2\lambda a(t) + 1]y_{t-} + \lambda^2 b(t)[\lambda b(t)d(t) + e(t)] = 0,\ a(T) = b(T) = c(T) = d(T) = e(T) = 0.$$

This means that the coefficients in $W^g(x_t, y_t, t)$, solve the following differential equations

(40) $a'(t) = -4\lambda^3 a(t)^3 - 2\lambda^2 a(t)^2 + [2\varphi + \rho]a(t),\ a(T) = 0$

(41) $b'(t) = -\gamma + [-4\lambda^3 2a(t)^2 + \varphi + \rho]b(t),\ \ b(T) = 0$

(42) $c'(t) = -\lambda^2 b(t)[\lambda a(t)b(t) + b(t)/2 + \lambda b(t)d(t) + e(t)] + \rho c(t),\ c(T) = 0$

(43) $d'(t) = [-4\lambda^2 a(t)d(t)\{\lambda a(t) + 1\} + 2\varphi + \rho]d(t),\ d(T) = 0$

(44) $e'(t) = \eta - 2\lambda^2 b(t)d(t)[2\lambda a(t) + 1] + [\varphi + \rho]e(t),\ e(T) = 0$

So $a(t) = d(t) = 0$ *and to* $u^*_{t-} = v^*_{t-} = b(t)$.

QED

We state without proof:

RESULT 2 (Saturating Incentives): *Suppose that the objectives are* $\int_0^T e^{-\rho t}[\gamma Max\{0, 1 - x_t^2\} - \eta y_t^2 - \frac{1}{2}u_t^2]dt$ *and* $\int_0^T e^{-\rho t}[\gamma Max\{0, 1 - y_t^2\} - \eta x_t^2 - \frac{1}{2}v_t^2]dt$ *such that incentives to invest saturate after the first arrival. In that case, the leader invests less than the laggard.*

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