Information exchange through secret vertical contracts *

Jihwan Do[†] and Nicolas Riquelme[‡]

February 1, 2022

Abstract

This paper studies a stylized common agency problem where two downstream firms, who operate in separated markets and receive private signals about a common demand state, simultaneously offer a secret menu of two-part tariffs contracts to their common supplier. While direct communication is not possible, they may still exchange their information through the use of signal-contingent menus of vertical contracts. We show that a perfect Bayesian equilibrium exists in which information is fully transmitted and the downstream firms obtain nearly first-best industry surplus. Our result suggests that, even in the presence of asymmetric information, efficient collusion with market allocation may not necessitate direct communication as long as firms trade with a common upstream supplier.

1 Introduction

In recent years, it is increasingly recognized that firms operating at the same level of a supply chain may exchange their strategic information via a common contractual partner operating at a different level of the supply chain. In one case, the UK competition authority found that there was indirect communication between Argos and Littelwoods (retailers in the UK toys market) via Hasbro (an upstream toy manufacturer in the UK), which constitutes a violation

*We thank Stefan Elbl for excellent research assistance on this project. We are grateful to Paulo Barelli, Yu Awaya, Pedro Jara-Moroni, Ana Elisa Pereira and audiences at University of Rochester, PUC-Chile, SECHI and SAET for helpful comments.

†Department of Economics, University of Rochester

‡The School of Business and Economics, Universidad de los Andes, Chile.

of antitrust laws. A similar antitrust case was brought against the manufacturer and retailers in the UK replica football kits markets.[1]

In their investigation into this type of collusion involving a common upstream supplier, competition authorities typically seek for the evidence of hub-and-spoke communication—a pair of bilateral communications between a manufacturer (hub) and each retailer (spoke)—having an impact on the degree of horizontal competition. While such back-and-forth bilateral communications might be sufficient to enforce efficient collusion, it remains unclear as to their necessity in that firms' actions taken in the vertical contracting stage may already convey relevant private information from each party. This way of indirectly exchanging information through vertical contracting may present a particularly attractive option to colluding firms when more direct communication is highly risky for antitrust scrutiny.[2]

The purpose of this paper is to explore such an alternative mechanism for privately informed downstream firms to exchange their information without communication. To do so, we analyze a stylized common agency model where two downstream firms, who operate in separated markets and receive private signals about a common demand state, simultaneously offer a secret menu of two-part tariffs contracts to their common upstream supplier. While direct communication is not possible, downstream firms may still exchange their information through the use of signal-contingent menus of vertical contracts. This is achieved if (i) different menus are offered by different types of a downstream firm (i.e., signaling) and (ii) different contracts in a menu are selected by different types of the upstream firm (i.e., screening)—where the upstream firm's type refers to which menu it is offered by another downstream firm.

The model we analyze is static without direct competitive externalities between downstream firms: each downstream firm operates in a single period as a local monopolist in one of two separated markets. This is the simplest environment in which the issue of indirect information exchange between downstream firms is separated from the issue of a cartel enforcement mechanism. In mapping our analysis onto collusive practices, hence, we interpret that downstream firms have already achieved an enforceable market allocation scheme, whereas information exchange without communication remains as a main problem for efficient outcomes. In practice, such an enforcement could be achieved by either vertical restraints such as exclusive territories or repeated interactions allowing for within-cartel punishments (unless monitoring for breaching such agreements is not possible).

In this framework, we show that there exists a perfect Bayesian equilibrium in which information is fully transmitted and each downstream firm obtains nearly first-best industry surplus. This result may appear obvious if communication is feasible in that each downstream firm operates as a local monopolist in each market. On the other hand, this is no longer immediate in our model where the only available device for information exchange is a secret menu of two-

[1] We refer to Harrington (2018) for more substantive examples.

[2] For vertical contracts to reflect each party's demand forecast is usually regarded as a regular business practice.

part tariffs contracts. Since firms' actions taken in the vertical contracting stage have payoff-relevant consequences, downstream firms might lack incentives to offer a "correct" menu of contracts and the upstream firm might want to choose a "wrong" contract. In particular, downstream firms can never obtain nearly first-best industry profits unless necessary distortions of contracts for incentive compatibility are sufficiently small.

The key assumption behind our result is the correlation of downstream firms' signals, which follows from that these signals are informative about a common demand state. This allows us to construct a menu of two-part tariffs contracts that are slightly perturbed from the menu containing only the first-best contract (maximizing downstream profits) without violating incentive compatibility conditions on either party.[3] In the equilibrium, each downstream firm learns that the other downstream firm receives a high (resp. low) signal when the manufacturer chooses a contract with a negative (resp. positive) fixed fee paid from a downstream firm. Thus, our result also suggests that there can be an informational value to the use of slotting fees—the cost manufacturers to pay for shelf space access.

This paper proceeds as follows: in Section 2, we provide a review of related literature; in Section 3, we present the framework; in Section 4, the main result is derived; in Section 5, we discuss some implications of our result and extensions; in Section 6, we conclude.

2 Related Literature

This paper is related to the literature on vertical contracting and segmented market collusion. Since we consider indirect information exchange via a common supplier, we also contribute to the literature on hub-and-spoke collusion.

Hub-and-spoke collusion has received much of attention recently since it was discovered as a common practice in many cartels and there is no clear understanding on the legal side to prosecute this behavior. A detailed study of multiple cases of hub-and-spoke collusion in different industries is given by Harrington (2018), who emphasizes that the hub played a fundamental role in coordinating and enabling the cartel to prosper. Odudu (2011) delves in detail about the legal issues that arise due to the current definitions of anti-trust entities towards collusive behaviour in European entities. The paper indicates that, although horizontal information sharing is commonly frowned upon, the regulations for vertical information sharing are less clear cut.

An intend to formalize theoretically hub-and-spoke collusion was done on Sahuguet & Walckiers (2017). In a Bertrand competition repeated game model, they show that a collusion scheme where retailers share information through the supplier can be sustained. Shamir (2017)

[3] Although a similar idea was proposed in Crémer & McLean (1988), our result is not a direct application of Crémer & McLean (1988) since we analyze a common agency model in which multiple informed principals trade with a common strategic agent through secret menus of two-part tariffs contracts.

analyzes a similar setting, with N different competing retailers and a common supplier among them. He concludes that retailers have the incentive to truthfully use cheap talk messages to transmit information to the supplier to ensure that the wholesale price is set in the most convenient manner according to all the available information of the retailers. This relies on the idea that the belief that the manufacturer has of the state of the demand is dependent on the messages he receives from the retailers, which skews his decision process in order to account for all of the information that is made available to him. Chu (1992) studies an inverse setup, where the supplier is the one that holds the private information about the level of demand of a new product. The information is then signaled by the manufacturer via his wholesale price and the retailer screens the veracity of the information by issuing a take-it-or-leave-it offer. The paper conclude that retailers may be able to infer the private information the supplier possesses via the contract that they offer to the supplier. A notion that motivates one of the backbones of this present paper.

Gilo & Yehezkel (2020) study secret vertical contracting with a common supplier with exchange of information through cheap talk and conclude that including a supplier in the collusive scheme increases the possibilities for it to prosper. In a similar setup, Shamir (2012) explores the transmission of information from the supplier to the retailers through the wholesale price. This leads to the retailers inferring the information of all the other retailers that are involved in the collusion. The paper is a first attempt to understand the possibility of bypassing the need to explicitly communicate amongst those involves, and in turn, diminish the possibility of them being prosecuted by antitrust authorities. Our project intend to fully understand this possibility considering exchange of information in both ways. In a model without informational issues, Marx & Shaffer (2007) shows that in equilibrium a retailer can use upfront payments to exclude its rival. Miklos-Thal, Rey & Vergé (2011) shows that this type of exclusion appears only when one retailers' contracts can not be contingent on the relationship being exclusive or not.

The literature on separated markets collusion is scarce. The seminal work Brander & Krugman (1983) shows in a Cournot model that lowering the cost for firms to compete in two markets improves welfare since reduces the monopolistic distortion in each individual market. Pinto (1986) expands upon that model and shows that cooperation among the firms is most fragile when it is more desirable. That is, the lower the trading costs, the less likely is for the cartel to subsist. Bernheim & Whinston (1990) study the impact of multi-market contact on the competitiveness of the markets as a whole. They conclude that under identical markets, firms and technology of constant returns to scale, the multi-market contact does not enhance the possibility of firms to sustain collusive prices. Bond & Syropoulos (2008) elaborate further on the multi-market setup and show that under sufficiently low discount factors and cost of selling in both markets, the efficient collusive agreements require requires firms to have presence in all of the possible markets. Something which contrasted with our result, that suggested that the best idea would be to allocate a single firm to each of the possible markets.

We contribute to the common agency literature. Applications are extensive and considers antitrust regulation, tacit collusion, multilateral lobbying and vertical contracting among many others (Bernheim, 1986). Galperti (2015) considers a general mechanism design framework where each principal has private information and provides a delegation principle type of result useful to characterize equilibrium allocations. Martimort and Moreira (2010) and Lima and Moreira (2014) consider specific common-agency games with privately informed principals. We contribute to this literature on showing how principals can use the common manufacturer to exchange relevant information to coordinate on a collusive outcome in the future. We also contribute to the informed principal mechanism design literature (Myerson, 1983; Maskin and Tirole, 1990; Maskin and Tirole, 1992; Mylovanov and Tröger, 2012). Here, in contrast, we consider multiple privately informed principals that choose mechanisms strategically to signal information that allows particular agent's information. In our construction, principals signal his own information through the design of a mechanism while at the same time screens agents information.

3 Preliminaries

3.1 Framework

We consider a setting in which there are two downstream firms D_1 and D_2 and one upstream supplier U. Downstream firm D_i sells the goods supplied by U in market M_i with $M_1 \neq M_2$. In order to supply q units of the goods, U incurs production costs $c(q) = c \cdot q$ with $c > 0$, whereas downstream costs are normalized to zero.

The aggregate demand is subject to stochastic shock denoted by $\theta \in \Theta = \{H, L\}$ with prior $\nu = \Pr(\theta = H) \in (0, 1)$. Although it is unobservable, each downstream firm D_i receives a private signal $s_i \in \mathcal{S} = \{h, l\}$ informative of true demand state. To be more precise, the inverse-demand of market M_i is given linear:

$$P(q_i; \theta) = \theta - q_i$$

where $q_i \geq 0$ is the quantity sold in market M_i. We write

$$\mu(s, s'|\bar{\theta}) = \Pr(s_1 = s, s_2 = s'|\theta = \bar{\theta}) \in (0, 1)$$

for the joint distribution of signals conditional on $\theta = \bar{\theta}$. Also, unconditional distribution is denoted by

$$\mu(s_1, s_2) = \sum_{\theta \in \Theta} \nu(\theta)\mu(s_1, s_2|\theta)$$

and the conditional probability of s_{-i} given s_i is given

$$\mu(s|s') = \Pr(s_j = s|s_i = s')$$

Finally, the posterior on θ given a pair of signals is denoted by

$$\nu(s, s') = \Pr(\theta = H|s_1 = s, s_2 = s')$$

Throughout the paper, we impose several assumptions on the signal distribution. First, we focus on symmetric case: for each $s, s' \in \mathcal{S}$ and $\theta \in \Theta$,

$$\mu(s, s'|\theta) = \mu(s', s|\theta)$$

Next, we assume without loss of generality,

$$\mathrm{E}(\theta|h, h) > \mathrm{E}(\theta|h, l) = \mathrm{E}(\theta|l, h) > \mathrm{E}(\theta|l, l)$$

In particular, this implies that for each $s, s' \in \mathcal{S}$,

$$\mathrm{E}(\theta|s, s') \neq \mathrm{E}(\theta|s)$$

which shows that the information exchange has *positive* value: the expected value of demand state can be different when a firm is informed of an additional signal on top of his own. We assume that $0 \leq c < \mathrm{E}(\theta|l, l)$ so that it is always efficient to produce even in the lowest demand estimate. Also $w \in [0, \mathrm{E}(\theta|l, l))$ is imposed, which guarantees that the optimal quantity is always nonnegative for any w and belief.

An important condition we will exploit is that signals are correlated. Remark that this is natural assumption given that two signals are informative about *common* demand states.[4]

Assumption 1. $\mu(h|h)\mu(l|l) \neq \mu(h|l)\mu(l|h)$

Now, we explain the protocol of vertical contracts. First, we assume that downstream firms have bargaining power and offer take-or-leave-it *menu* of contracts to the manufacturer. In particular, a contract is restricted to *two-part tariffs* denoted by (w, T), where $w \in \mathbb{R}_+$ is a linear wholesale price and $T \in \mathbb{R}$ is fixed payment. Remark that negative fixed payment corresponds to a slotting fee paid by manufacturer to retailer. We denote a menu offered by D_i conditional on his own signal s by

$$\{(w_{i,l}^s, T_{i,l}^s), (w_{i,h}^s, T_{i,h}^s)\}$$

A typical contract is written by $d_i = (w_i, T_i) \in \mathbb{R}_+ \times \mathbb{R} \equiv \Delta_i$ and the default contract is denoted by $d_0 = (0, 0)$. A menu of contracts is then given by $m_i \in \Delta_i^2 \times \{d_0\} \equiv \Delta \times \{d_0\} \equiv M_i$.

[4]As an illustration, consider following information structure: $\Pr(\theta = H) = 1/2$ and $\Pr(s_i = h|\theta = H) = \Pr(s_i = l|\theta = L) = \rho > 1/2$, where s_1 and s_2 are iid conditional on θ.

Given an offer of menu m_i, the manufacturer may accept one of the two non-default contracts or pick the default one, which corresponds to rejecting the offered menu. A contract offer and acceptance decision between U and D_i are not observable to D_{-i}.

Note that U's payoff is the aggregate profits he obtains from each contracts. More precisely, if U has accepted contract $d_i = (w_i, T_i)$ with downstream firm D_i, who in turn orders q_i units, then his profit from bilateral contract with D_i is given by

$$u_i(q_i, d_i) = (w_i - c)q_i + T_i$$

Thus, the upstream supplier's total profits is given by

$$u(q_1, q_2, d_1, d_2) = \sum_{i=1,2} u_i(q_i, d_i) = \sum_{i=1,2} (w_i q_i + T_i)$$

while downstream firm D_i obtains

$$\pi(q_i, d_i; \theta) = (P(q_i; \theta) - w_i)q_i - T_i = (\theta - q_i - w_i)q_i - T_i$$

Remark that total bilateral surplus in a pair $U - D_i$ only depends on q_i, which is written by

$$v(q_i; \theta) = u_i(q_i, d_i) + \pi(q_i; \theta) = (\theta - q_i - c)q_i$$

An expected surplus conditional on $(s_1, s_2) = (s, s')$ is denoted by

$$\begin{aligned} v(q; s, s') &= \mathrm{E}\left(v(q_i; \theta) | s, s'\right) = \mathrm{E}\left(\theta - q - c)q | s, s'\right) \\ &= (\mathrm{E}(\theta | s, s') - q - c)q \end{aligned}$$

Letting $q^M(s, s') = \arg\max_{q \geq 0} v(q; s, s') = (\mathrm{E}(\theta | s, s') - c)/2$, the first-best bilateral surplus is given by

$$v^M = \sum_{(s,s') \in \mathcal{S}^2} \mu(s, s') \cdot \mathrm{E}\left(v(q^M(s, s'); \theta) | s, s'\right) = \sum_{(s,s') \in \mathcal{S}^2} \mu(s, s') \left(\frac{(\mathrm{E}(\theta | s, s') - c)^2}{4} \right)$$

We summarize the timing of the game in Figure 1,

3.2 Equilibrium

Our equilibrium concept is perfect Bayesian equilibrium in pure strategies. More formally, a strategy of downstream firm D_i consists of (i) a function $\sigma_{m,i} : \mathcal{S} \to M_i$ mapping a signal to a menu of contracts; and (ii) a function $\sigma_{q,i} : \mathcal{S} \times M_i \times \Delta_i \to \mathbb{R}_+$, a mapping from his own signal, a offer of contract menu, and a choice of contract to a quantity order. A strategy for the

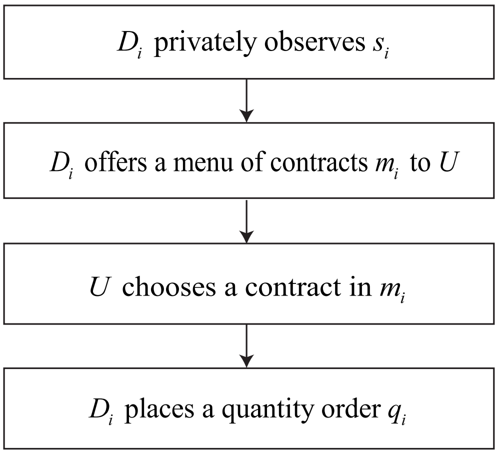

Figure 1: sequence of events

upstream supplier U consists of a function $\sigma_U : M_1 \times M_2 \to \Delta_1 \times \Delta_2$ mapping from a pair of contract menus to acceptance decisions, where its i-th coordinate is written by $\sigma_{U,i}$. Finally, a typical strategy profile will be denoted by $\sigma = (\sigma_U, \sigma_{m,i}, \sigma_{q,i})_{i=1,2}$.

To define a perfect Bayesian equilibrium, we need to specify a belief system. First, note that the upstream supplier updates his belief about demand state θ and signal pairs s_1, s_2 when observing observing a pair of menu offers. We write U's posterior by

$$\beta_U(\theta, s_1, s_2 | m_1, m_2)$$

and its marginal on $s_i \in \mathcal{S}$ will be denoted by

$$\beta_U(s_i | m_1, m_2) = \sum_{(\theta, s_{-i}) \in \Theta \times \mathcal{S}} \beta_U(\theta, (s_i; s_{-i}))$$

On the other hand, each downstream firm updates his belief when (i) receiving a private signal and (ii) observing which contract is accepted by the upstream firm. More precisely, let $\beta_i(\theta, s_{-i} | s_i)$ denote D_i's posterior after signal realization s_i and $\beta_i(s_{-i}|s_i)$ be its marginal on $s_{-i} \in \mathcal{S}$. Similarly,

$$\beta_i(\theta, s_{-i} | s_i, m_i, d_i)$$

refers to the posterior of D_i after a history (s_i, m_i, d_i), and the marginal on $\theta \in \Theta$ is denoted by $\beta_i(\theta | s_i, m_i, d_i)$. Finally, a typical belief system will be denoted by $\beta = (\beta_U, (\beta_i)_{i=1,2})$.

Note that a pair of strategy profile and belief system (σ, β) forms a perfect Bayesian equilibrium (PBE) if σ is *sequentially rational* given β and β is *Bayesian consistent* with σ. More precisely, σ is called sequentially rational given β if:

(i) $\sigma_{m,i}$ is optimal for D_i:

$$\sigma_{m,i}(s_i) \in \arg\max_{m_i \in M_i} \mathrm{E}_{\beta_i} \left(\pi(\sigma_{q,i}, \sigma_{U,i}; \theta) | s_i \right)$$

(ii) $\sigma_{q,i}$ is optimal for D_i:

$$\sigma_{q,i}(s_i, m_i, d_i) \in \arg\max_{q_i \in \mathbb{R}_+} \mathrm{E}_{\beta_i}\left(\pi(q_i, d_i; \theta)|s_i, m_i, d_i\right)$$

(iii) σ_U is optimal for U:

$$\sigma_U(m_1, m_2) \in \arg\max_{(d_1,d_2)\in m_1\times m_2} \mathrm{E}_{\beta_U}\left(u(\sigma_{q,1}, \sigma_{q,2}, d_1, d_2)|m_1, m_2\right)$$

A belief system β is called Bayesian consistent with σ if β is obtained from Bayes rule whenever possible:

(i) for every on-path menus $m = (m_1, m_2) \in \sigma_{m,1}(\mathcal{S}) \times \sigma_{m,2}(\mathcal{S})$,

$$\beta_U(H, s_1, s_2|m_1, m_2) = \nu(s_1, s_2)\left(\frac{\mu(s_1, s_2)}{\mu(\text{supp }(m))}\mathbb{1}_{\text{supp }(m)}(s_1, s_2)\right)$$

where supp $(m) = \{(s_1, s_2) \in \mathcal{S}^2 : (\sigma_{m,1}(s_1), \sigma_{m,2}(s_2)) = m\}$ and $\mu(\text{supp }(m)) = \sum_{(s_1,s_2)\in\text{supp }(m)} \mu(s_1, s_2)$.

(ii) for every $s_i \in \mathcal{S}$, $m_i = \sigma_{m,i}(s_i)$ and $d_i \in \sigma_{U,i}(m_i, \sigma_{m,-i}(\mathcal{S}))$,

$$\beta_i(H, s_{-i}|s_i, m_i, d_i) = v(s_i, s_{-i})\left(\frac{\mu(s_{-i}|s_i)}{\mu(\text{supp }(d_i|m_i)|s_i)}\mathbb{1}_{\text{supp }(d_i|m_i)}(s_{-i})\right)$$

where supp $(d_i|m_i) = \{s_{-i} \in \mathcal{S} : \sigma_{U,i}(m_i, \sigma_{m,i}(s_{-i})) = d_i\}$ and $\mu(\text{supp }(d_i|m_i)|s_i) = \sum_{s_{-i}\in\text{supp }(d_i|s_i)} \mu(s_{-i}|s_i)$.

Remark that there is no restriction on off-path beliefs.

4 Main result

In this section, we present our main result: vertical contracts in the form of two-part tariffs can replace direct horizontal communication. In addition, the cost of such indirect communication can be arbitrarily small.

First, remark that necessary conditions for downstream firms to achieve nearly first-best profits include the followings: first, information should be exchanged, meaning that *signaling* and *screening* should occur at the same time; second, an accepted contract must include a unit wholesale price w that has *no* significant difference from c. Otherwise, each downstream firm does not internalize production costs incurred to U, which might reduce bilateral surplus they can achieve. Finally, the fixed payment T from D_i to U should not be so large.

In the followings, we impose the assumptions that the value of information is sufficiently high so that if a firm with a demand signal h (or l resp.) can obtain additional signal for *free*, then he does not find it profitable to pretend to receive l (or h resp.). More precisely,

Assumption 2 (Sufficient value of information sharing)**.** There exists $\hat{s} \in \mathcal{S}$ such that

$$\sum_{s' \in \mathcal{S}} \mu(s'|\hat{s}) \frac{\left(E_{v^{\hat{s}}_{s'}}(\theta) - c\right)^2}{4} > \max_{w \in \mathbb{R}_+} \left(\frac{(E_{v^{\hat{s}}}(\theta) - w)^2}{4} + \frac{(E_{v^{-\hat{s}}}(\theta) - w)\,(w - c)}{2} \right)$$

This assumption is used only for constructing an off-path belief to deter a particular type deviation, which manipulates the upstream firm's belief by payoff-irrelevant contract in a menu. Remark that it is *sufficient* condition, not *necessary*, so one may find an alternative off-path belief system to support on-path plays without imposing such parametric assumptions. Nonetheless, it greatly simplifies non-trivial issues of off-path deviations of menus, and is satisfied in a broad range of parameters as the following example shows:

Example 1. Suppose $H = 3, L = 1, \nu = 1/2$. In addition, s_1 and s_2 are i.i.d. conditional on θ, with precision $Pr(s_i = h|\theta = H) = Pr(s_i = l|\theta = L) = \rho \in (1/2, 1)$. For each $c \in (0, 1)$, there exists $\bar{\rho}(c) \in (1/2, 1)$ such that Assumption 2 is satisfied for all $r > \bar{\rho}(c)$. In particular, $\bar{\rho}(c)$ converges to $1/2$ as c goes to 0.

Now, we are ready to state our main result:

Theorem 1. Suppose Assumption 2 is satisfied. Given $\epsilon > 0$, there exists a PBE in which information is fully exchanged, and each downstream firm obtains at least $v^M - \epsilon$..

Proof. Here, we provide the proof of Theorem 1. To do this, let us write

$$m^s = \{d^s_l, d^s_h, d_0\} = \{(w^s_l, T^s_l), (w^s_h, T^s_h), d_0\}$$

for the candidate equilibrium menu offered by each firm D_i receiving signal $s \in \mathcal{S}$. Also, given a linear wholesale price $w \geq 0$ and a belief β that $\theta = H$, we denote the optimal quantity level at the last stage by

$$q^*(w, \beta) = \arg\max_{q \in \mathbb{R}_+} \mathrm{E}_\beta(P(q; \theta) - w)q = \frac{\mathrm{E}_\beta(\theta) - w}{2}$$

In addition, downstream firm's expected profit will be written by

$$\pi(w, \beta) = \mathrm{E}_\beta\left(P(q^*(w, \beta); \theta) - w)q^*(w, \beta)\right) = \frac{(\mathrm{E}_\beta(\theta) - w)^2}{4}$$

Henceforth, we further simplify the notation for the posterior on demand state by writing $\nu(s, s') = \nu^s_{s'} = \nu^{s'}_s$. We first ignore off-path behaviors and focus on the on-path incentive compatibility conditions.

Let $\epsilon > 0$ be given. We construct a pair of menus (m^l, m^h) that depend on ϵ such that $m^l \neq m^h$ and a firm that receives signal s prefers to offer menu m^s. We refer to these restrictions as incentive compatibility conditions for signaling. At the same time, we require that for each pair of menus (m^s, m^{-s}) the manufacturer prefers to pick contracts $d^s_{-s} \in m^s$ and $d^{-s}_s \in m^{-s}$. We refer to these restrictions as incentive compatibility conditions for screening. In this way, in the construction, information is fully exchanged. Moreover, each downstream firm obtains $v^M - \epsilon$. The construction holds regardless of Assumption 2.

As a first step, let us summarize the on-path incentive compatibility conditions. Note that each menu can screen the upstream supplier when

$$T^s_l + q^*(w^s_l, \nu^s_l)(w^s_l - c) \geq T^s_h + q^*(w^s_h, \nu^s_h)(w^s_h - c)$$
$$T^s_h + q^*(w^s_h, \nu^s_h)(w^s_h - c) \geq T^s_l + q^*(w^s_l, \nu^s_l)(w^s_l - c)$$

From these two inequalities, we have

$$u^s \equiv T^s_l + q^*(w^s_l, \nu^s_l)(w^s_l - c) = T^s_h + q^*(w^s_h, \nu^s_h)(w^s_h - c)$$

Notice that U's participation constraint—picking a non-default contract—is simply satisfied if $u^s \geq 0$ for each $s \in \mathcal{S}$. Using this condition, we rewrite the fixed payments as follows:

$$T^s_l = u^s - q^*(w^s_l, \nu^s_l)(w^s_l - c)$$
$$T^s_h = u^s - q^*(w^s_h, \nu^s_h)(w^s_h - c)$$

Then, each downstream firm is willing to signal his type if for each $s \neq -s \in \mathcal{S}$,

$$\sum_{s' \in \mathcal{S}} \mu(s'|s) \left[\pi(w^s_{s'}, \nu^s_{s'}) + q^*(w^s_{s'}, \nu^s_{s'})(w^s_{s'} - c)\right] - u^s$$
$$\geq \sum_{s' \in \mathcal{S}} \mu(s'|s) \left[\pi(w^{-s}_{s'}, \nu^s_{s'}) + q^*(w^{-s}_{s'}, \nu^{-s}_{s'})(w^{-s}_{s'} - c)\right] - u^{-s}$$

or equivalently,

$$u^s - u^{-s} \leq \sum_{s' \in \mathcal{S}} \mu(s'|s) \Delta^s_{s'}$$

where

$$\Delta^s_{s'} = \left(\pi(w^s_{s'}, \nu^s_{s'}) + q^*(w^s_{s'}, \nu^s_{s'})(w^s_{s'} - c) - \pi(w^{-s}_{s'}, \nu^s_{s'}) - q^*(w^{-s}_{s'}, \nu^{-s}_{s'})(w^{-s}_{s'} - c)\right)$$

Therefore, incentive compatibility conditions for signaling is summarized by

$$-\sum_{s' \in \mathcal{S}} \mu(s'|l) \Delta^l_{s'} \leq u^h - u^l \leq \sum_{s' \in \mathcal{S}} \mu(s'|h) \Delta^h_{s'} \tag{1}$$

Note that

$$\begin{aligned}
\Delta^s_{s'} &= \mathrm{E}_{\nu^s_{s'}} \left(\left(P\left(q^*\left(w^s_{s'}, \nu^s_{s'}\right); \theta\right) - c\right) q^*\left(w^s_{s'}, \nu^s_{s'}\right) - \left(P\left(q^*\left(w^{-s}_{s'}, \nu^s_{s'}\right); \theta\right) - c\right) q^*\left(w^{-s}_{s'}, \nu^s_{s'}\right) \right) \\
&\quad + \left(q^*\left(w^{-s}_{s'}, \nu^s_{s'}\right) - q^*\left(w^{-s}_{s'}, \nu^{-s}_{s'}\right)\right)\left(w^{-s} - c\right) \\
&= \left(\mathrm{E}_{\nu^s_{s'}}(\theta) - q^*\left(w^s_{s'}, \nu^s_{s'}\right) - c\right) q^*\left(w^s_{s'}, \nu^s_{s'}\right) - \left(\mathrm{E}_{\nu^s_{s'}}(\theta) - q^*\left(w^{-s}_{s'}, \nu^s_{s'}\right) - c\right) q^*\left(w^{-s}_{s'}, \nu^s_{s'}\right) \\
&\quad + \left(q^*(w^{-s}_{s'}, \nu^s_{s'}) - q^*\left(w^{-s}_{s'}, \nu^{-s}_{s'}\right)\right)\left(w^{-s}_{s'} - c\right)
\end{aligned}$$

Now, consider following class of linear wholesale prices parameterized by $\eta > 0$:

$$w_{s'}^{s} = c + \eta \cdot \alpha(s')$$

where $\alpha(s')$ will be defined below. Under these unit prices, we have

$$\begin{aligned}\Delta_{s'}^{s} &= \left(q^{*}\left(w_{s'}^{-s}, \nu_{s'}^{s}\right) - q^{*}\left(w_{s'}^{-s}, \nu_{s'}^{-s}\right)\right)\left(w_{s'}^{-s} - c\right) \\ &= \eta \cdot \left(\frac{\mathrm{E}_{\nu_{s'}^{s}}(\theta) - \mathrm{E}_{\nu_{s'}^{-s}}(\theta)}{2}\right)\alpha(s') = -\Delta_{s'}^{-s}\end{aligned}$$

Therefore, the right-hand side of (4) can be written as

$$\begin{aligned}\mu(h|h)\Delta_{h}^{h} + \mu(l|h)\Delta_{l}^{h} &= \eta \cdot \left(\mu(h|h)\left(\frac{\mathrm{E}_{\nu_{h}^{h}}(\theta) - \mathrm{E}_{\nu_{h}^{l}}(\theta)}{2}\right)\alpha(h) + \mu(l|h)\left(\frac{\mathrm{E}_{\nu_{l}^{h}}(\theta) - \mathrm{E}_{\nu_{l}^{l}}(\theta)}{2}\right)\alpha(l)\right) \\ &= \begin{pmatrix}\mu(h|h) & \mu(l|h)\end{pmatrix} \times \begin{pmatrix}\xi_{h} \cdot \alpha(h) \cdot \eta \\ \xi_{l} \cdot \alpha(l) \cdot \eta\end{pmatrix}\end{aligned}$$

where

$$\xi_{s'} = \frac{\mathrm{E}_{\nu_{s'}^{h}}(\theta) - \mathrm{E}_{\nu_{s'}^{l}}(\theta)}{2} > 0$$

Similarly, the left-hand side of (4) equals to

$$-\left(\mu(h|l)\Delta_{h}^{l} + \mu(l|l)\Delta_{l}^{l}\right) = \begin{pmatrix}\mu(h|l) & \mu(l|l)\end{pmatrix} \times \begin{pmatrix}\xi_{h} \cdot \alpha(h) \cdot \eta \\ \xi_{l} \cdot \alpha(l) \cdot \eta\end{pmatrix}$$

Now, consider following system of linear equations:

$$\begin{aligned}&\begin{pmatrix}\eta \\ -\eta\end{pmatrix} = \begin{pmatrix}\mu(h|h) & \mu(l|h) \\ \mu(h|l) & \mu(l|l)\end{pmatrix} \times \begin{pmatrix}\xi_{h} \cdot \alpha(h) \cdot \eta \\ \xi_{l} \cdot \alpha(l) \cdot \eta\end{pmatrix} \\ \iff &\begin{pmatrix}1 \\ -1\end{pmatrix} = \begin{pmatrix}\mu(h|h) & \mu(l|h) \\ \mu(h|l) & \mu(l|l)\end{pmatrix} \times \begin{pmatrix}\xi_{h} \cdot \alpha(h) \\ \xi_{l} \cdot \alpha(l)\end{pmatrix}\end{aligned}$$

Letting $\rho \equiv \mu(h|h)\mu(l|l) - \mu(h|l)\mu(l|h) \neq 0$, the unique solution $(\alpha^{*}(h), \alpha^{*}(l))$ is given by

$$\begin{aligned}\alpha^{*}(h) &= \frac{\mu(l|l) + \mu(l|h)}{\rho \cdot \xi_{h}} \\ \alpha^{*}(l) &= -\frac{\mu(h|l) + \mu(h|h)}{\rho \cdot \xi_{l}}\end{aligned}$$

Note that when $\alpha^{*}(h)$ and $\alpha^{*}(l)$ are chosen, the left-hand side of (4) becomes $-\eta$ while the right-hand side becomes η. In particular, the condition can be clearly satisfied with $u^{h} = u^{l} = 0$.

The remaining task is to show that each menu contains two different contracts and two menus are different. First, note that each menu contains two contracts whose wholesale prices are different: $\alpha^*(h) > 0 > \alpha^*(l)$ if $\rho > 0$ while $\alpha^*(h) < 0 < \alpha^*(l)$ if $\rho < 0$. To check that two menus are different, note that

$$\begin{aligned} T_h^h - T_h^l &= -q^*(w_h^h, \nu_h^h)(w_h^h - c) + q^*(w_h^l, \nu_h^l)(w_h^l - c) \\ &= \left(q^*(c + \eta\alpha^*(h), \nu_h^l) - q^*(c + \eta\alpha^*(h), \nu_h^h)\right)\eta\alpha^*(h) \end{aligned}$$

which is strictly negative (resp. positive) if $\rho > 0$ (resp. $\rho < 0$). Therefore, two menus are clearly distinguished.

Finally, note that $\alpha^*(h)$ and $\alpha^*(l)$ are independent of $\eta > 0$. This implies that $w_{s'}^s$ converge to c when η becomes arbitrarily close to 0. In turn, the fixed payments also tend to 0 as η converges to 0. As a result, each downstream firm obtains the profit arbitrarly close to v^M when η is small enough.

To complete the proof of the theorem, we need to show that the Nash equilibrium in the lemma is indeed a perfect Bayesian equilibrium. To do this, we now construct off-path beliefs and strategies, assuming without loss that Assumption 2 holds with $\hat{s} = h$. More precisely, consider an arbitrary off-path menu offered from D_i, denoted by

$$m^0 = \{(w_1^0, T_1^0), (w_2^0, T_2^0), d_0\}$$

Recall that $\beta_U(s_i|m^0, m_{-i})$ denotes U's belief about s_i. Letting $\bar{w}^0 = \max\{w_1^0, w_2^0\}$, we assign

$$\beta_U(h|m^0, m_{-i}) \equiv \beta_U(m^0) = \begin{cases} 0 & \text{if } \bar{w}^0 \geq c \\ 1 & \text{if } \bar{w}^0 < c \end{cases}$$

for any $m_{-i} \in M_i$. Similar to U's on-path incentive compatibility condition, menu m^0 can screen the upstream firm only when U is indifferent:

$$\begin{aligned} &T_1^0 + \left(\beta_U(m^0)q^*(w_1^0, \nu^h) + (1 - \beta_U(m^0))q^*(w_1^0, \nu^l)\right)(w_1^0 - c) \\ &= T_2^0 + \left(\beta_U(m^0)q^*(w_2^0, \nu^h) + (1 - \beta_U(m^0))q^*(w_2^0, \nu^l)\right)(w_2^0 - c) \end{aligned}$$

where $\nu^s = \Pr(\theta = H|s_i = s)$ is the posterior conditional on a single signal. Thus, it is sequentially rational for U to choose a contract in m^0 with higher unit wholesale price regardless of the offer m_{-i} whenever indifferent. In particular, this implies that a downstream firm cannot learn the other downstream firm's signal when offering any off-path menu m^0.

Given this belief, U is willing to accept a contract (w_i^0, T_i^0) if

$$\begin{aligned} &T_i^0 + \left(\beta_U(m^0)q^*(w_i^0, \nu^h) + (1 - \beta_U(m^0))q^*(w_i^0, \nu^l)\right)(w_i^0 - c) \geq 0 \\ \iff &-T_i^0 \leq \left(\beta_U(m^0)q^*(w_i^0, \nu^h) + (1 - \beta_U(m^0))q^*(w_i^0, \nu^l)\right)(w_i^0 - c) \end{aligned}$$

Thus, an off-path contract (w_i^0, T_i^0) provides a profit to deviating downstream firm at most

$$\begin{aligned}
&\mathrm{E}_{\nu^s}\left(\left(P\left(q^*\left(w_i^0,\nu^s\right);\theta\right)-w_i^0\right)q^*\left(w^0,\nu^s\right)\right)\\
&+\left(\beta_U(m^0)q^*(w_i^0,\nu^h)+(1-\beta_U(m^0))q^*(w_i^0,\nu^l)\right)(w_i^0-c)\\
=&\frac{(\mathrm{E}_{\nu^s}(\theta)-w_i^0)^2}{4}+\left(\frac{\beta_U(m^0)\mathrm{E}_{\nu^h}(\theta)+(1-\beta_U(m^0))\mathrm{E}_{\nu^l}(\theta)-w_i^0}{2}\right)(w_i^0-c)
\end{aligned}$$

We can check that the first-order condition is satisfied at

$$w_i^0 = c - \mathrm{E}_{\nu^s}(\theta) + \beta_U(m^0)\mathrm{E}_{\nu^h} + (1-\beta_U(m^0))\mathrm{E}_{\nu^l}(\theta) \tag{2}$$

Suppose now that a downstream firm with signal h offers off-path menu m^0. There are two cases considered: (i) $\bar{w}^0 < c$ and (ii) $\bar{w}^0 \geq c$. In the first case, U forms a belief $\beta_U(m^0) = 1$, and so (2) implies that the upper bound on a deviation profit is given by

$$\frac{(\mathrm{E}_{\nu^h}(\theta)-c)^2}{4}$$

On the other hand, equilibrium offer provides a payoff arbitrarily close to

$$\sum_{s'\in\mathcal{S}}\mu(s'|h)\left(\frac{(\mathrm{E}_{\nu^h_{s'}}(\theta)-c)^2}{4}\right)$$

which, by Jensen's ienquality,

$$\begin{aligned}
\sum_{s'\in\mathcal{S}}\mu(s'|h)\left(\frac{(\mathrm{E}_{\nu^h_{s'}}(\theta)-c)^2}{4}\right) &> \frac{\left(\sum_{s'\in\mathcal{S}}\mu(s'|h)\mathrm{E}_{\nu^h_{s'}}(\theta)-c\right)^2}{4}\\
&= \frac{(\mathrm{E}_{\nu^h}(\theta)-c)^2}{4}
\end{aligned}$$

On the other hand, we have $\beta_U(m^0) = 0$ if $\bar{w}^0 \geq c$. Certainly, if an accepted contract calls for wholesale price higher or equal to c, then (2) implies that the upper bound on deviation profit is achieved at lower boundary $\bar{w}^0 = c$, which is given

$$\frac{(\mathrm{E}_{\nu^h}(\theta)-c)^2}{4}$$

By the same logic applied to case (i), such deviations cannot be profitable either. The remaining case is that a deviating menu contains one contract with wholesale price below c, and another with wholesale price greater than or equal to c. However, such a deviation is not profitable either by Assumption 2.

Next, consider a deviation by a downstream firm with signal l. If m^0 is offered and $\bar{w}^0 < c$, then $\beta_U(m^0) = 1$. Again, (2) implies that the upper bound on deviation profit, which is achieved at upper boundary $\bar{w}^0 = c$, is given

$$\frac{(\mathrm{E}_{\nu^l}(\theta)-c)^2}{4}$$

On the other hand, equilibrium offer provides a payoff arbitrarily close to

$$\sum_{s' \in \mathcal{S}} \mu(s'|l) \left(\frac{(\mathrm{E}_{\nu^l_{s'}}(\theta) - c)^2}{4} \right)$$

which, by Jensen's ienquality,

$$\begin{aligned} \sum_{s' \in \mathcal{S}} \mu(s'|l) \left(\frac{(\mathrm{E}_{\nu^l_{s'}}(\theta) - c)^2}{4} \right) &> \frac{\left(\sum_{s' \in \mathcal{S}} \mu(s'|l) \mathrm{E}_{\nu^l_{s'}}(\theta) - c \right)^2}{4} \\ &= \frac{(\mathrm{E}_{\nu^l}(\theta) - c)^2}{4} \end{aligned}$$

Finally, note that $\beta_U(m^0) = 0$ when $\bar{w}^0 \geq c$. Then, (2) implies that the upper bound on deviation profit is again given by

$$\frac{(\mathrm{E}_{\nu^l}(\theta) - c)^2}{4}$$

By the same logic above, such deviations cannot be profitable either. □

Our equilibrium provides a potential explanation to the use of fixed charges versus slotting fees in vertical contracting. Our explanation provides from screening manufacturer's information. In our model a retailer offers a menu that includes a contract with a slotting fee and a high wholesale price (compared to the cost) to screen other retailer's signal h and uses a fixed fee and a low wholesale price (compared to the cost) to screen other retailer's signal l. These provides the manufacturer the right incentives to pick a contract and transmitting information: reveal a high signal from picking a high wholesale price and reveal a low signal from picking a contract with a fixed fee.

In the equilibrium, the menus are different only in the up-front amount of money: transfers are higher in absolute terms (fixed and slotting fee) in the retailer type h menu. This allows retailers to transmit their information to the manufacturer. Retailer's incentives to offer the right menu depends on the expected transfer to receive (or give) conditional on screening. Retailer's signal affects how each retailer weights the fixed and slotting fee. A type h retailer put more relative weight to the difference in the slotting fees to receive compared to the difference in the fixed fees to pay, while a type l retailer is the opposite. Retailers payoff from the market is not relevant for transmission of information: all the information is transmitted using different transfers. The net payoff obtained from production does not depends on manufacturers beliefs directly. Thus, it is not relevant at the time to see incentive compatibility conditions for the retailers.

5 Discussion

5.1 Alternative market allocations

We have assumed that each retailer is a local monopoly, which may be achieved by vertical restraints such as exclusive territory clauses. Alternatively, such market allocation schemes can also be self-enforcing when firms interact repeatedly. In particular, Theorem 1 suggests that the only information necessary to implement the nearly perfect collusion is whether some firm has stolen the market assigned to another. In other words, firms have no incentive to deviate in the contracting stage so long as they believe that the market allocation agreement will be respected. If this information—whether each firm has refrained from supplying in the other's market—is made public and firms are sufficiently patient, retailers could sustain nearly efficient collusion by various punishment mechanisms such as the grim-trigger or carrot-and-stick strategies.

Still, one may wonder if the similar result will arise even when firms are competing, provided that signals are correlated. However, this turns out to be not the case as shown below. Therefore, it reveals the additional superiority of market allocation rule in terms of communication cost, which could be an explanation for why such market allocation schemes are widely employed in real-world cartels.

More specifically, we continue to assume that both downstream firms operate in two markets, but unlike to our main model, each firm is allowed to sell positive quantity in any market. The inverse-demand function with total quantity $q_1 + q_2$ is given by

$$P(q_1 + q_2; \theta) = \theta - (q_1 + q_2)$$

where θ is stochastic demand shock. A menu offered by a downstream firm and accepted contract is not observable to the opponent firm. Our benchmark is the case when there is complete information. In this case, both retailers and manufacturer observe both retailer's signals. Denote retailer's payoff in the benchmark case as v^B.

We focus on perfect Bayesian equilibria in which contracting strategies are symmetric: that is, (i) a menu of contracts offered D_i is equal to that of D_{-i} whenever two firms received a same signal, and (ii) U's acceptance decision is symmetric: for each $(m_1, m_2) \in M$,

$$(\sigma_{U,1}(m_1, m_2), \sigma_{U,2}(m_1, m_2)) = (\sigma_{U,2}(m_2, m_1), \sigma_{U,1}(m_2, m_1))$$

In this case, a retailer may pretend to have received a different signal than the real one and then produce strategically in the competition stage. For example, a potential deviation for a retailer could be to signal to the manufacturer to have received signal l, so then the manufacturer transmits this information to the other retailer, making him less optimistic about the demand. The previous decreases other retailer's expected marginal revenue, making him produce less in equilibrium. This allows the retailer to increase its marginal revenue, producing more in equilibrium.

In the next proposition we show that the equilibrium payoff for retailers when every signal is public is not achievable by any equilibrium with private signals.

Proposition 1. There exists $\kappa > 0$ such that each downstream firm's ex-ante profits in any equilibrium where information is fully exchanged is no greater than $v^B - \kappa$.

Proof. See Appendix. $\square$

Intuitively, private information adds an extra inefficiency to the one produced by the competition between retailers. This justifies our primary focus on a market allocation where the demand is split between the two retailers, and the only inefficiency comes from dispersed information.

5.2 Extension: many signals

In this section, we show that our result can be extended beyond binary signal cases. Suppose that the set of signals is $\mathcal{S} = \{s_1, s_2, ..., s_n\}$, in which for any $i > i'$,

$$\nu(s_i, s_j) = \nu(s_j, s_i) > \nu(s_i', s_j) = \nu(s_j, s_i').$$

Also, we assume that the following $n \times n$ symmetric matrix E is invertible:

$$E \equiv \begin{pmatrix} \mathrm{E}_{\nu_{s_1}^{s_1}}(\theta) & \cdots & \mathrm{E}_{\nu_{s_1}^{s_n}}(\theta) \\ \cdots & \cdots & \cdots \\ \mathrm{E}_{\nu_{s_n}^{s_1}}(\theta) & \cdots & \mathrm{E}_{\nu_{s_n}^{s_n}}(\theta) \end{pmatrix}$$

The main result of this section is the following.

Proposition 2. Suppose that for each $s \in \mathcal{S}$, there exists no $(\lambda(-s))_{-s \neq s}$ such that (i) $\lambda(-s) \geq 0$ for all $-s$ and (ii) $\mu(s'|s) = \sum_{-s \neq s} \lambda(-s)\, \mu(s'|-s)$ for all $s' \in S$. Then, given $\epsilon > 0$, there exists a Nash equilibrium in which information is fully exchanged, and each downstream firm obtains at least $v^M - \epsilon$.

Proof. See Appendix. $\square$

Note that the condition in the proposition also appears in Cremer and Mclean (1989), and our result is reminiscent of theirs. Unlike Cremer and Mclean (1989), however, the current model is a common agency game in which each principal (retailer) holds ex-ante private information which is transmitted in equilibrium through secret contracting with the common agent (manufacturer). Thus, our construction involves not only screening as in Cremer and Mclean (1989), but also signaling from each retailer to their common manufacturer. In addition, the set of feasible contracts in our setting is restricted to two-part tariff contracts, which significantly differs from Cremer and Mclean (1989) who consider general transfer schemes.

To obtain more intuition about our result, we consider an alternative assumption under which our result holds. Define for this case

$$\xi^s_{-s}(s') = \frac{\mathrm{E}_{\nu^s_{s'}}(\theta) - \mathrm{E}_{\nu^{-s}_{s'}}(\theta)}{2},$$

and

$$\Delta^s_{-s}(s') = \left(\pi(w^s_{s'}, \nu^s_{s'}) + q^*(w^s_{s'}, \nu^s_{s'})(w^s_{s'} - c) - \pi(w^{-s}_{s'}, \nu^s_{s'}) - q^*(w^{-s}_{s'}, \nu^{-s}_{s'})(w^{-s}_{s'} - c)\right).$$

Incentive compatibility conditions for types s and $-s$ are summarized by

$$-\sum_{s' \in \mathcal{S}} \mu(s'|-s)\Delta^{-s}_s(s') \leq 0 \leq \sum_{s' \in \mathcal{S}} \mu(s'|s)\Delta^s_{-s}(s') \tag{3}$$

Let's change the notation from s' to s_j to make evident the index of each type of signal. Suppose $w^s_{s_j} = c + \eta\alpha(j) := w_{s_j}$. Thus

$$\Delta^s_{-s}(s_j) = \eta\alpha(j)\left[q^*(w_{s_j}, \nu^s_{s_j}) - q^*(w_{s_j}, \nu^{-s}_{s_j})\right] = \eta\alpha(j)\xi^s_{-s}(s_j)$$

and then Equation 3 becomes

$$\sum_{j=1}^{n} \mu(s_j|-s)\alpha(s_j)\xi^s_{-s}(s_j) \leq 0 \leq \sum_{j=1}^{n} \mu(s_j|s)\alpha(s_j)\xi^s_{-s}(s_j).$$

Denote as F the set

$$F = \left\{\alpha : \{1, \cdots, n\} \to \mathbb{R} : \alpha(j)\xi^s_{-s}(j) \text{ increases in } j, \text{for every } -s < s\right\}$$

The set F is not empty. For example, we can consider $\alpha(1) = -1$ and $\alpha(j) = (j-1)\frac{1}{n-1}$ for every other j and assume that $\xi^s_{-s}(j)$ is increasing in j. For a fixed function $\alpha \in F$, define the set of probability distributions $D^s_{-s}(\alpha) = \left\{\nu \in \Delta^{n-1} : \sum_{j=1}^n \nu_j\alpha(s_j)\xi^s_{-s}(s_j) = 0\right\}$. Note that D^s_{-s} is a convex set. Suppose that $\mu(\cdot|s)$ increases in a first order stochastic dominance sense. Thus, for Equation 3 to hold is sufficient the existence of a distribution $\nu_0 \in D^s_{-s}(\alpha)$ such that

$$\mu(\cdot|-s) \preceq_{fosd} \nu_0 \preceq_{fosd} \mu(\cdot|s).$$

Thus, the following assumption is also sufficient for our result.

Assumption 3. There are distributions $\nu^j_i \in D^j_i(\alpha)$ for every $i < j$ for some $\alpha \in F$ such that

$$\mu(\cdot|s_i) \preceq_{fosd} \mu^j_i \preceq_{fosd} \mu(\cdot|s_j).$$

5.3 Antitrust implications

According US antitrust law, a hub-and-spoke conspiracy is used to characterize competitors agreements to eliminate competition that include participants who are in the vertical relationship. The main feature of it is the participation of the vertically conspirator in the horizontal agreement. It is *per se* illegal for horizontal competitors to collude, whether on their own or through an intermediary.

There are many legitimate vertical arrangements and restraints that firms use in their relationships with upstream or downstream partners.[5] However, these types of restraints may also be used for the organization of a cartel.

To prove a hub-and-spoke conspiracy, An horizontal agreement between retailers is necessary to prove in a hub-and-spoke conspiracy. There must be "direct or circumstantial evidence that reasonably tends to prove" an agreement. Courts generally look for circumstantial evidence or factors—that may allow them to infer an horizontal agreement. These factors include: the spokes acting against self-interest; spokes knowing about agreements with other spokes and expecting reciprocity; abrupt changes to business practices; bid rigging among spokes; communication among spokes; or communications from hubs to spokes regarding other spokes' intentions. Because the hub has a legitimate business relationship with each spoke, communications between the hub and a spoke may not attract antitrust scrutiny. [6]

We provide theoretical evidence that firms can exchanging information without using any direct communication and instead only the necessary contracting between business partners. This result suggest many collusion cases that can not be prosecuted in court. We should advance research on this type of collusion which allows this special type of exchange information. Adjusting antitrust law to accommodate this possibility seems necessary.

5.4 Increasing number of firms

Our results relies on the existence of a common manufacturer between two firms to be able to exchange information. In case the number of manufacturers that retailers contract is more than one, retailer can always replicate our construction with each manufacturer and obtains the same result. In case the number of retailers is more than two, from the perspective of one retailer, the design of the optimal menu of contracts becomes a multidimensional screening problem, which makes the number of contracts necessary to encapsulate every possible information from other retailers to be large. However, we believe our construction can be used to construct such a menu of contracts, at a cost of complicate the notation. An important feature

[5]Geographic restraints, exclusivity clauses, resale price maintenance (RPM), most favored nation clauses, loyalty discounts, and similar types of restraints can all be used for legitimate business purposes independent of forming a cartel or engaging in a conspiracy

[6]http://www.oecd.org/daf/competition/hub-and-spoke-arrangements.htm

from our model that allows our construction is that manufacturer's payoff depends indirectly on retailer's information, so screening requires to make the manufacturer indifferent between different contracts. The same feature can be leverage to screen in the more general case.

6 Conclusion

In this paper, we have explored indirect information exchanges between downstream firms via secret vertical contracts with a common upstream manufacturer. In doing so, we developed a stylized model of a common agency game in which two downstream firms with private demand signals, who operate in separate markets, trade with their common upstream manufacturer. We restrict feasible contracts to those of two-part tariffs and assume that a vertical contract of a pair of upstream and downstream firm is not observable to the other downstream firm.

We showed that there exists an equilibrium in which information is fully exchanged and the downstream firms extract nearly perfect first-best surplus. A notable feature of this equilibrium is the informational value of using slotting fees—the cost manufacturers to pay for shelf space access. When the correlation of two signals are positive, for instance, each downstream firm may learn that the other downstream firm receives a high (resp. low) signal when the manufacturer chooses a contract with a negative (resp. positive) fixed fee paid from a downstream firm.

Our finding has relevant implications on competition policies, which put strong emphasis on direct communication among colluding firms: efficient collusion among downstream firms may not necessitate direct and explicit communication, even though information they hold is decentralized. Once downstream firms have successfully divided the market, they may stop a horizontal exchange of information and instead rely on a common manufacturer to transmit information in hub-and-spoke ways, even if vertical contracts remain secret to each other.

There are several important but challenging issues that we leave for future research. First, one may ask the scope of indirect information exchange when downstream firms are competing within the same market. Though we provided a brief discussion on this case (Proposition 1), our analysis is incomplete in that we only considered implementing efficient outcomes. Second, we analyzed the simplest environment where the market demand and cost functions are linear. In particular, the assumption of constant marginal cost has greatly simplified our analysis because it rules out downstream firms' competition in input markets. Relaxing this assumption will be another interesting future work.

References

Crémer, J. and McLean, R. (1988). Full extraction of the surplus in Bayesian and dominant strategy auctions. *Econometrica: Journal of the Econometric Society*, 1247-1257.

Athey, S., & Bagwell, K. (2001). Optimal collusion with private information. *RAND Journal of Economics*, 428-465.

Athey, S., Bagwell, K., & Sanchirico, C. (2004). Collusion and price rigidity. *The Review of Economic Studies*, 71(2), 317-349.

Bernheim, B. D., & Whinston, M. D. (1986). Common agency. *Econometrica*, 54 (4) (1986) 923–942.

Bernheim, B. D., & Whinston, M. D. (1990). Multimarket contact and collusive behavior. *The RAND Journal of Economics*, 1-26.

Bond, E. W., & Syropoulos, C. (2008). Trade costs and multimarket collusion. *The RAND Journal of Economics*, 39(4), 1080-1104.

Brander, J., & Krugman, P. (1983). A 'reciprocal dumping'model of international trade. *Journal of International Economics*, 15(3-4), 313-321.

Chu, W. (1992). Demand signalling and screening in channels of distribution. *Marketing Science*, 11(4), 327-347.

Ciliberto, F., & Williams, J. W. (2014). Does multimarket contact facilitate tacit collusion? Inference on conduct parameters in the airline industry. *The RAND Journal of Economics*, 45(4), 764-791.

Galperti, S. (2015). Common agency with informed principals: Menus and signals. *Journal of Economic Theory*, 157, 648-667.

Gerlach, H. (2009). Stochastic market sharing, partial communication and collusion. *International Journal of Industrial Organization*, 27(6), 655-666.

Gilo, D., & Yehezkel, Y. (2020). Vertical collusion. *The RAND Journal of Economics*, 51(1), 133-157.

Harrington Jr, J. E. (2018). How Do Hub-and-Spoke Cartels Operate? Lessons from Nine Case Studies. Lessons from Nine Case Studies (August 24, 2018).

Lima, R.C. & Moreira, H. (2014). Information transmission and inefficient lobbying. *Games and Economic Behavior*, 86 (2014) 282–307.

Martimort, D. & Moreira, H. (2010). Common agency and public good provision under asymmetric information. *Theoretical Economics*, 5 (2) (2010) 159–213.

Marx, L. M., & Shaffer, G. (2007). Upfront payments and exclusion in downstream markets. *The RAND Journal of Economics*, 38(3), 823-843.

Maskin, E., & Tirole, J. (1990). The principal-agent relationship with an informed principal: The case of private values *Econometrica*, 379-409.

Maskin, E., Tirole, J. (1992). The principal-agent relationship with an informed principal, II: Common values. *Econometrica*, 1-42.

Miklos-Thal, J., Rey, P., & Vergé, T. (2011). Buyer Power and Intraband Coordination. *Journal of the European Economic Association*, vol. 9 (nº 4). pp. 721-741.

Myerson, R. B. (1983). Mechanism design by an informed principal. *Econometrica*, 1767-1797.

Mylovanov, T., & Tröger, T. (2012). Informed-principal problems in environments with generalized private values. *Theoretical Economics*, 7(3), 465-488.

Odudu, O. (2011). Indirect information exchange: the constituent elements of hub and spoke collusion. *European Competition Journal*, 7(2), 205-242.

Pinto, B. (1986). Repeated games and the 'reciprocal dumping'model of trade. *Journal of International Economics*, 20(3-4), 357-366.

Röller, L. H., & Steen, F. (2006). On the workings of a cartel: Evidence from the Norwegian cement industry. *American Economic Review*, 96(1), 321-338.

Sahuguet, N., & Walckiers, A. (2017). A theory of hub-and-spoke collusion. *International Journal of Industrial Organization*, 53, 353-370.

Shamir, N. (2017). Cartel formation through strategic information leakage in a distribution channel. *Marketing Science*, 36(1), 70-88.

Shamir, N. (2012). Strategic information sharing between competing retailers in a supply chain with endogenous wholesale price. *International Journal of Production Economics*, 136(2), 352-365.

Supplementary Information for Information Exchange through Secret Vertical Contracts

7 Proofs

7.1 Proof of Proposition 1

First we solve the complete information case equilibrium. Suppose retailers choose contracts simultaneously. Suppose retailers observe signals i and j. Letting q_i and q_j denote the equilibrium quantities of D_i and D_j respectively, the following condition should be satisfied:

$$q_i = \arg\max_q \left(\mathrm{E}\left(\theta|i,j\right) - q_j - q - w_i\right) q$$

Solving these, we obtain the equilibrium quantities:

$$\begin{aligned} q_i &= \frac{\mathrm{E}\left(\theta|i,j\right) - 2w_i + w_j}{3} \\ q_j &= \frac{\mathrm{E}\left(\theta|i,j\right) - 2w_j + w_i}{3} \end{aligned}$$

At the beginning of the game, each retailer forms beliefs about the contract the other retailer signs with the manufacturer. Note that here there is no loss of generality to assume that each retailer will offer a contract instead of a menu to the manufacturer. There is no information a retailer can obtain from the manufacturer. Thus, each retailer solves:

$$\begin{aligned} &\max_{w_i,T_i} q_i(w_i,w_j)^2 + T_i \\ \text{st: } &(w_i - c)q_i(w_i,w_j) - T_i \geq 0 \end{aligned}$$

In the optimum, retailer i makes the manufacturer indifferent. The the problem becomes:

$$\max_{w_i} q_i(w_i,w_j)^2 + (w_i - c)q_i(w_i,w_j)$$

From the first order conditions we obtain $2q_i\frac{-2}{3} + q_i + (w_i - c)\frac{-2}{3} = 0$ and thus $q_i = 2(c - w_i)$. This implies decreasing best responses functions for the retailer. In equilibrium

$$w_i^* = w_j^* = \frac{6c - \mathrm{E}\left(\theta|i,j\right)}{5}$$

Note that $w_i < c$. Moreover, in equilibrium, retailer i's payoff increases with w_j.

Pick a pair w_i, w_j.

Consider now the case when signals are privately observed. Suppose that there exists an equilibrium where signaling and screening occur, where a menu offered by D_i with signal s is

$$m^s = \{d_l^s, d_h^s, d_0\} = \{\left(w_l^s, T_l^s\right), \left(w_h^s, T_h^s\right), d_0\}$$

Let us write $\hat{w} = (w_h^h, w_l^h, w_h^l, w_l^l)$ for simplicity. Suppose that D_i's signal is s, and D_{-i}'s signal is s'. Then, D_i will offer m^s, and D_{-i} will offer $m^{s'}$. Accordingly, U will accept the contract $d^s_{s'}$ offered by D_i, and accept the contract d^s_s offered by D_{-i}. Letting $q^*(s,s')$ and $q^*(s',s)$ denote the equilibrium quantities of D_i and D_{-i} respectively, the following condition should be satisfied:[7]

$$\begin{aligned} q^*(s,s') &= \arg\max_q \left(\mathrm{E}\left(\theta|s,s'\right) - q^*(s',s) - q - w^s_{s'}\right) q \\ &= \frac{\mathrm{E}\left(\theta|s,s'\right) - q^*(s',s) - w^s_{s'}}{2} \end{aligned}$$

Solving these, we obtain the equilibrium quantities, $\{q^*(s,s') : s,s' \in \mathcal{S}\}$:

$$\begin{aligned} q^*(h,h) &= \frac{\mathrm{E}\left(\theta|h,h\right) - w^h_h}{3} \\ q^*(h,l) &= \frac{\mathrm{E}\left(\theta|h,l\right) - 2w^h_l + w^l_h}{3} \\ q^*(l,h) &= \frac{\mathrm{E}\left(\theta|h,l\right) - 2w^l_h + w^h_l}{3} \\ q^*(l,l) &= \frac{\mathrm{E}\left(\theta|l,l\right) - w^l_l}{3} \end{aligned}$$

Now, consider U's incentive compatibility conditions. As in our previous section, we can easily check that U is willing to choose the right contract in a menu m^s if and only if

$$u^s \equiv T^s_h + q^*(s,h)(w^s_h - c) = T^s_l + q^*(s,l)(w^s_l - c)$$

for each $s \in \mathcal{S}$. Certainly, U's participation constraint is given by $u^s \geq 0$, and we can represent fixed paymnents by

$$T^s_h = u^s - q^*(s,h)(w^s_h - c) \tag{4}$$
$$T^s_l = u^s - q^*(s,l)(w^s_l - c) \tag{5}$$

Consider D_i with signal h. Notice that the incentive compatibiltiy requires D_i to have no incentive for double-deviation. In particular, it should prevent D_i from offering m^l and choosing $q^*(h,s')$ after U choose contract $d^l_{s'} \in m^l$:

$$\begin{aligned} &\sum_{s' \in \mathcal{S}} \mu(s'|h) \left(\left(\mathrm{E}\left(\theta|h,s'\right) - q^*(s',h) - q^*(h,s') - w^h_{s'}\right) q^*(h,s') - T^h_{s'}\right) \\ \geq &\sum_{s' \in \mathcal{S}} \mu(s'|h) \left(\left(\mathrm{E}\left(\theta|h,s'\right) - q^*(s',l) - q^*(h,s') - w^l_{s'}\right) q^*(h,s') - T^l_{s'}\right) \end{aligned}$$

Using the expression (3) and (4), one can check that this inequality is equivalent to

$$u^l - u^h \geq \sum_{s' \in \mathcal{S}} \mu(s'|h) \left(\left(q^*(s',h) - q^*(s',l)\right) q^*(h,s') + \left(q^*(h,s') - q^*(l,s')\right)\left(c - w^l_{s'}\right)\right)$$

[7]Given that the contracting strategies are symmetric, it is straightforward that the optimal quantities are also symmetric across downstream firms.

Since $u^h \geq 0$, this implies that

$$u^l \geq \sum_{s' \in \mathcal{S}} \mu(s'|h) \left(\left(q^*(s',h) - q^*(s',l) \right) q^*(h,s') + \left(q^*(h,s') - q^*(l,s') \right) \left(c - w^l_{s'} \right) \right)$$

If we replicate the contracts from the complete information case here, the right hand side would be strictly positive and then $u^l > 0$. Thus, retailers would be forced to leave informational rents to the manufacturer in order to implement the same contracts than the complete information case.

Let's consider now the highest possible payoff a retailer can obtain from an equilibrium with information transmission. From the perspective of a retailer type s, his optimal menu m^s must be different from m^{-s}. Moreover, the contracts in m^s must be such U's incentive compatibility conditions are satisfied. These impose extra constraints compared to the maximization problem in the complete information case. Since the optimum in the complete information case gives to the retailer strictly lower payoff in the present case, the payoff obtained in the equilibrium with highest payoff is strictly lower. This completes the proof.

7.2 Proof of Proposition 2

Denote

$$m^s = \{d^s_{s_1}, \ldots, d^s_{s_n}, d_0\} = \{(w^s_{s_1}, T^s_1), \ldots, (w^s_n, T^s_n), d_0\}$$

for the equilibrium menu offered by each firm D_i receiving signal $s \in \mathcal{S}$. Each menu can screen the upstream supplier when for all s, s', and s'',

$$T^s_{s'} - q^*\left(w^s_{s'}, \nu^s_{s'}\right)\left(w^s_{s'} - c\right) \geq T^s_{s''} - q^*\left(w^s_{s''}, \nu^s_{s''}\right)\left(w^s_{s''} - c\right)$$

Thus, the upstream supplier's (IC) and (IR) conditions will be satisfied if for all s and s',

$$u^s \equiv T^s_{s'} - q^*\left(w^s_{s'}, \nu^s_{s'}\right)\left(w^s_{s'} - c\right) \geq 0$$

As in the binary signal case, let us consider the following class of contract menus that induce $u^s = 0$ for all $s \in \mathcal{S}$:

$$w^s_{s'} = c + \eta\alpha\left(s'\right) \equiv w_{s'}$$

and

$$T^s_{s'} = q^*\left(w^s_{s'}, \nu^s_{s'}\right)\left(w^s_{s'} - c\right) = \eta\alpha\left(s'\right) q^*\left(w_{s'}, \nu^s_{s'}\right)$$

Note that a retailer of type s has an incentive to offer m^s if for all $-s \neq s$,

$$\begin{aligned} & \sum_{s' \in \mathcal{S}} \mu\left(s'|s\right) q^*\left(w_{s'}, \nu^s_{s'}\right) \eta\alpha\left(s'\right) > \sum_{s' \in \mathcal{S}} \mu\left(s'|s\right) q^*\left(w_{s'}, \nu^{-s}_{s'}\right) \eta\alpha\left(s'\right) \\ \Longleftrightarrow & \sum_{s' \in \mathcal{S}} \mu\left(s'|s\right) q^*\left(w_{s'}, \nu^s_{s'}\right) \alpha\left(s'\right) > \sum_{s' \in \mathcal{S}} \mu\left(s'|s\right) q^*\left(w_{s'}, \nu^{-s}_{s'}\right) \alpha\left(s'\right) \end{aligned}$$

Since $q^*(w,\nu) = \frac{\mathrm{E}_\nu(\theta)-w}{2}$, this inequality is equivalent to

$$\sum_{s'\in\mathcal{S}} \mu(s'|s)\,\alpha(s')\,\mathrm{E}_{\nu^s_{s'}}(\theta) > \sum_{s'\in\mathcal{S}} \mu(s'|s)\,\alpha(s')\,\mathrm{E}_{\nu^{-s}_{s'}}(\theta)$$

Letting $\lambda(s) = \left(\alpha(s_1)\,\mathrm{E}_{\nu^s_{s_1}}(\theta), \alpha(s_2)\,\mathrm{E}_{\nu^s_{s_2}}(\theta), \cdots, \alpha(s_n)\,\mathrm{E}_{\nu^s_{s_n}}(\theta)\right)^T$, we can rewrite as follows:

$$\mu(\cdot|s)\cdot\lambda(s) > \mu(\cdot|s)\cdot\lambda(-s)$$

for all $s \neq -s$, where $\mu(\cdot|s) = (\mu(s_1|s), \cdots, \mu(s_n|s))^T$.

Letting A^s be the $n\times(n-1)$ matrix defined by

$$A^s = \begin{pmatrix} \mu(s_1|s_1) & \mu(s_1|s_2) & \cdots & \mu(s_1|s-1) & \mu(s_1|s+1) & \cdots & \mu(s_1|n) \\ \mu(s_2|s_1) & \mu(s_2|s_2) & \cdots & \mu(s_2|s-1) & \mu(s_2|s+1) & \cdots & \mu(s_2|n) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu(s_n|s_1) & \mu(s_n|s_2) & \cdots & \mu(s_n|s-1) & \mu(s_n|s+1) & \cdots & \mu(s_n|n) \end{pmatrix}$$

and b^s be the n-dimensional vector

$$b^s = (\mu(s_1|s), \mu(s_2|s), \cdots, \mu(s_n|s))^T,$$

Assumption 3 says that there exists no $x^s \in \mathbb{R}^{n-1}_+$ such that $A^s x^s = b^s$. By Farkas lemma, this implies that there is $y^s \in \mathbb{R}^n_+$ such that

$$b^s\cdot y^s = \mu(s_1|s)\,y^s_1 + \cdots + \mu(s_n|s)\,y^s_n > 0$$

and

$$(A^s)^T y^s = \begin{pmatrix} \mu(s_1|s_1) & \mu(s_2|s_1) & \cdots & \mu(s_n|s_1) \\ \cdots & \cdots & \cdots & \cdots \\ \mu(s_1|s_n) & \mu(s_2|s_n) & \cdots & \mu(s_n|s_n) \end{pmatrix} \begin{pmatrix} y^s_1 \\ y^s_2 \\ \cdots \\ y^s_n \end{pmatrix}$$

$$= \begin{pmatrix} \mu(s_1|s_1)y^s_1 + \mu(s_2|s_1)y^s_2 + \cdots + \mu(s_n|s_1)y^s_n \\ \cdots \\ \mu(s_1|s-1)y^s_1 + \mu(s_2|s-1)y^s_2 + \cdots + \mu(s_n|s-1)y^s_n \\ \mu(s_1|s+1)y^s_1 + \mu(s_2|s+1)y^s_2 + \cdots + \mu(s_n|s+1)y^s_n \\ \cdots \\ \mu(s_1|s_n)y^s_1 + \mu(s_2|s_n)y^s_2 + \cdots + \mu(s_n|s_n)y^s_n \end{pmatrix} \leq \begin{pmatrix} 0 \\ \cdots \\ 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

Let Y be the $n\times n$ matrix defined by

$$Y = \begin{pmatrix} y^1_1 & y^2_1 & \cdots & y^n_1 \\ \cdots & \cdots & \cdots & \cdots \\ y^1_n & y^2_n & \cdots & y^n_n \end{pmatrix}$$

Note that

$$
\begin{aligned}
&\Big(\lambda(s_1);\lambda(s_2);\cdots;\lambda(s_n)\Big)\\
&=\begin{pmatrix}
\alpha(s_1)\mathrm{E}_{\nu_{s_1}^{s_1}}(\theta) & \alpha(s_1)\mathrm{E}_{\nu_{s_1}^{s_2}}(\theta) & \cdots & \alpha(s_1)\mathrm{E}_{\nu_{s_1}^{s_n}}(\theta)\\
\alpha(s_2)\mathrm{E}_{\nu_{s_2}^{s_1}}(\theta) & \alpha(s_2)\mathrm{E}_{\nu_{s_2}^{s_2}}(\theta) & \cdots & \alpha(s_2)\mathrm{E}_{\nu_{s_2}^{s_n}}(\theta)\\
\cdots & \cdots & \cdots & \cdots\\
\alpha(s_n)\mathrm{E}_{\nu_{s_n}^{s_1}}(\theta) & \alpha(s_n)\mathrm{E}_{\nu_{s_n}^{s_2}}(\theta) & \cdots & \alpha(s_n)\mathrm{E}_{\nu_{s_n}^{s_n}}(\theta)
\end{pmatrix}\\
&=\begin{pmatrix}
\alpha(s_1) & 0 & 0 & \cdots & 0\\
0 & \alpha(s_2) & 0 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \cdots & \alpha(s_n)
\end{pmatrix}
\begin{pmatrix}
E_{\nu_{s_1}^{s_1}}(\theta) & \cdots & E_{\nu_{s_1}^{s_n}}(\theta)\\
\cdots & \cdots & \cdots\\
E_{\nu_{s_n}^{s_1}}(\theta) & \cdots & E_{\nu_{s_n}^{s_n}}(\theta)
\end{pmatrix}
\end{aligned}
$$

If $\Big(\lambda(s_1);\lambda(s_2);\cdots;\lambda(s_n)\Big)=Y$, then (IC)'s are satisfied since

$$\mu(\cdot|s)\cdot\lambda(s)=\mu(\cdot|s)\cdot(y_1^s,\cdots y_n^s)^T>0$$

and for all $-s\neq s$,

$$\mu(\cdot|s)\cdot\lambda(-s)=\mu(\cdot|s)\cdot\left(y_1^{-s},\cdots,y_n^{-s}\right)^T\leq 0$$

Thus, it suffices to show the existence of $\alpha(s_i)$'s such that

$$
Y=\begin{pmatrix}
\alpha(s_1) & 0 & 0 & \cdots & 0\\
0 & \alpha(s_2) & 0 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \cdots & \alpha(s_n)
\end{pmatrix}
\begin{pmatrix}
\mathrm{E}_{\nu_{s_1}^{s_1}}(\theta) & \cdots & \mathrm{E}_{\nu_{s_1}^{s_n}}(\theta)\\
\cdots & \cdots & \cdots\\
\mathrm{E}_{\nu_{s_n}^{s_1}}(\theta) & \cdots & \mathrm{E}_{\nu_{s_n}^{s_n}}(\theta)
\end{pmatrix}
$$

Clearly, if the symmetric matrix

$$
E\equiv\begin{pmatrix}
\mathrm{E}_{\nu_{s_1}^{s_1}}(\theta) & \cdots & \mathrm{E}_{\nu_{s_1}^{s_n}}(\theta)\\
\cdots & \cdots & \cdots\\
\mathrm{E}_{\nu_{s_n}^{s_1}}(\theta) & \cdots & \mathrm{E}_{\nu_{s_n}^{s_n}}(\theta)
\end{pmatrix}
$$

is invertible, there exist $\alpha(s_i)$'s satisfying the inequalities. In addition, we can always perturb the constructed vector $\alpha(\cdot)$ so that $\alpha(s_i)\neq\alpha(s_j)$ for $i\neq j$ since (IC) inequalities are strict.