The Impact of Institutions on Social Polarization

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Abstract

Representing social institutions in terms of the well-known DeGroot model of information transmission and social learning, the paper shows the ability of institutions to facilitate or prevent social consensus. It is also shown that the disconnectedness of institutions from the social network has a profound impact on the time required to achieve consensus.

Keywords: DeGroot model, reducible Markov chains, social consensus, social institutions, social polarization

1. Introduction

The term, "institution" is somewhat unclear both in ordinary language and in the academic literature. However, contemporary sociology is somewhat more consistent in its use of the term. Typically, contemporary sociologists use the term to refer to complex social forms that reproduce themselves such as governments, the family, human languages, universities, hospitals, business corporations, and legal systems.[1] Certain institutions play a central role in the sharing of information and opinion formation throughout society. Examples are the media, organized religion, political parties, the office of the President of the United States, etc. It is with these that this paper is concerned.

Fundamental questions are how social institutions influence whether or not the heterogeneous opinions of individuals converge to a common societal opinion and to what opinion they converge, if they do. A related question is, if society does reach a consensus, how do social institutions influence how long it takes. Representing social institutions within the framework of the well-known DeGroot model of information transmission and consensus formation, the research reported in this paper sheds light on these questions. The main contributions are Propositions 1 and 2 and Sections 3 and 7. Proposition 1 shows that in the case of a single institution, the institutional opinion becomes the consensus opinion. Proposition 2 shows that in the case of several institutions that disagree, the rest of society can reach a consensus only if all agents agree on the value of the institutional opinions. In Section 3, I define a structure called the *differencing matrix* to measure how far the network is from consensus or, more generally, the level of disagreement among agents. In Section 7, using a result on nearly uncoupled stochastic matrices, I show how institutional insularity - the disconnectedness of institutions from the rest of the network - affects the time required to achieve consensus.

The questions listed above have been of considerable interest in several fields and relevant results are scattered across the literatures of sociology, mathematics, economics, physics and computer science among others.[2] The line of research from which this paper springs began with French Jr. (1956) who introduced a simple model of how a network of interpersonal influence enters into the process of opinion formation. At each point in time, the members of a population simultaneously update their opinions to a value that is the *mean* of their own opinion and the opinions of those members who have directly communicated their opinions to them, where an opinion is represented by a single real number. French deduced that, over time, the opinions of members would converge to a single opinion, depending on the structure of influential communications in the population. Harary (1959) demonstrated the formal isomorphism between French's formalization and the theory of higher transition probabilities in Markov chains. Independently, and apparently unaware of the work of French and Harary, DeGroot (1974) considered the problem of a group of individuals that must act together as a committee, each of whom can specify his or her own subjective probability

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[1] Miller, Seumas, "Social Institutions", The Stanford Encyclopedia of Philosophy (Spring 2011 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/spr2011/entries/social-institutions/>.

[2] See Jackson (2010) for an overview and references.

distribution for the unknown value of a parameter of common interest. He presented a model which described how the group might reach a consensus and form a common subjective probability distribution for the parameter simply by revealing their individual distributions to each other and pooling their opinions. The model would also apply when the opinion of each member is represented simply as a point estimate rather than as a probability distribution. Thus the DeGroot model subsumed and extended the work of French and Harary.

In the DeGroot model, the social influence structure is represented as a matrix and that matrix remains constant throughout the updating process. By considering nonconstant updating rules, the DeGroot model was extended by other researchers. Using results from inhomogeneous Markov chain theory, Chatterjee and Seneta (1977) generalized the model to the case where the social influence matrix changes from update to update. Friedkin and Johnsen (1990) examined a model in which the updating process always puts some weight on an individual's initial beliefs. More recently, Krause (2000) presented a model in which an agent pays attention only to those whose opinions don't differ much from the agent's own. In the model of DeMarzo, Vayanos and Zwiebel (2003), an agent is allowed to change the weight he puts on his own beliefs over time. An excellent account of these developments and others can be found in chapter 8 of Jackson (2010).

The central theme of this paper is the influence that institutions have on societal consensus. In sociology, since the work of Katz (1953), French Jr. (1956) and Bonacich (1987), eigenvector-like notions of centrality and prestige have been put forward as measures of social influence. Because such models are often based on convergence of iterated influence relationships, the DeGroot model, in addition to providing an intuitive measure of social influence, also provides insight into the structure of the influence vectors in such models. However, consensus as defined in the DeGroot model, itself provides a natural measure of social influence. In DeGroot (1974), if a consensus is reached, the limiting beliefs are a weighted average of the initial beliefs. The social influence of each agent can then be defined as the relative weight each agent has on the final consensus beliefs. DeMarzo et al. (2003) show that social influence depends not only on the accuracy of one's beliefs, but also on how well connected one is in the social network in which communication takes place. Golub and Jackson (2010) focus on the situation where there is some true state of nature that agents are trying to learn and each agent's initial belief is equal to the true state plus noise. They call societies for which the consensus is the true value "wise." They show that a general obstacle to wisdom in arbitrary networks is the existence of prominent groups that receive a disproportionate share of attention. This dovetails neatly with the result in Section 5 that, in a social network, a single institution can determine the social consensus.

The paper is organized as follows. In Section 2 I introduce and explain the DeGroot model and cite a theorem from Golub and Jackson (2010) that states the necessary and sufficient conditions for convergence. In Section 3, expanding on an idea in DeMarzo et al. (2003), I introduce the differencing matrix as a way to measure differences of opinion. In preparation for the analysis of institutions, Appendix B briefly reviews primitive matrices and reducible Markov chains, and using the Perron-Frobenius theorem, states conditions under which a reducible Markov chain is convergent. Representing social networks with institutions as reducible Markov chains, Section 5 shows that in the case of a single institution, the institutional opinion becomes the consensus opinion. The case of the media vs. Sen. Joseph McCarthy is cited as an example. Defining a partial consensus as agreement among non-institutional agents, Section 6 uses the differencing matrix of Section 3 to derive the condition under which a partial consensus can be achieved in the presence of opposing institutions. It is seen that the improbability of the condition almost guarantees polarization between constituencies of the institutions among non-institutional agents. The gridlock between congressional Republicans and Democrats circa 2012 is used as an example. In Section 7, the effect of institutional insularity on time to consensus is examined with the conflict in Northern Ireland used to illustrate the discussion. Section 8 concludes.

The paper uses definitions and theorems from many different sources. Propositions 1 and 2, and Definition 3 are original. Results from journal articles are cited at the time they are used. Results not specifically cited are from Meyer (2000), a truly exceptional textbook on matrix analysis and linear algebra.

2. Social Influence: the DeGroot Model

The seminal interaction model of information transmission, opinion formation and consensus formation is due to DeGroot (1974). The model is simple, tractable and captures some aspects of social learning so it is unsurprising that it has a long history of precursors and reincarnations. It has been elucidated and extended by DeMarzo, Vayanos and Zwiebel (2003) and Golub and Jackson (2010) among others. An excellent discussion of the DeGroot model can be found in Jackson (2010).

Members of the network start with initial opinions (beliefs) represented by an n-dimensional vector $\mathbf{b}^{(0)} = \left[b_1^{(0)}, b_2^{(0)}, \ldots, b_n^{(0)}\right]^{\mathsf{T}}$. The member interactions are captured through an $n \times n$ nonnegative matrix $\mathbf{T}$ called the *social influence matrix*. In particular, let $\mathbf{T}$ be a (row) stochastic matrix so that its entries across each row sum to 1. The social influence matrix $\mathbf{T}$ is *convergent* if $\lim_{t\to\infty} \mathbf{T}^t \mathbf{b}^{(0)}$ exists for all initial vectors $\mathbf{b}^{(0)}$.

Agents update opinions by repeatedly taking weighted averages of their neighbors' opinions according to $\mathbf{T}$ where T_{ij} is the weight or trust that agent i places on agent j's opinion as he considers agent j's opinion in updating his own. That is, at the kth update, agents' opinions are given by

$$\mathbf{b}^{(k)} = \mathbf{T}\mathbf{b}^{(k-1)} = \mathbf{T}^k \mathbf{b}^{(0)}. \tag{1}$$

This process has the following motivation discussed by DeMarzo et al. (2003). At time $t = 0$ each agent receives a noisy signal $b_i^{(0)} = \mu + \varepsilon_i$, where $\varepsilon_i \in \mathbb{R}$ is a noise term with expectation 0 and μ is some state of nature. Agent i hears the opinions of the agents with whom he interacts and assigns precision π_{ij} to the opinion of agent j. If agent i does not listen to agent j, $\pi_{ij} = 0$. If the agents are Bayesian and the noise terms are normally distributed then agent i updates according to Eq. (1), setting $T_{ij} = \pi_{ij} / \sum_k \pi_{ik}$. At time $t = 2$ agents realize that their neighbors have new information (collected at $t = 1$) and so it is worthwhile to listen again to collect this new, indirect information. With multiple communication rounds, agents' information involves repetitions. Correctly adjusting for repetition would impose a heavy computational burden on the agents. The DeGroot model can be thought of as a boundedly rational version of this process where the agents do not adjust their weightings over time.

For example, suppose a public official has been accused of malfeasance and there are three other agents in the official's organization who are familiar with the situation. Let agents 1 and 3 both know agent 2 but not each other and let their initial estimates of the probability that the official is guilty be given by

$$\mathbf{b}^{(0)} = [1, 1/2, 0]^{\mathsf{T}}. \tag{2}$$

According to $\mathbf{b}^{(0)}$, agent 1 initially believes that the official is guilty with probability 1. Agent 2 initially believes there is a 50-50 chance that the official is innocent. Agent 3 on the other hand, initially believes that the official is definitely innocent. Suppose further that the three agents weight each others' opinions according to the matrix

$$\mathbf{T} = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix}. \tag{3}$$

Agent 1 places $3/4$ weight on his own opinion and $1/4$ on agent 2's opinion. He doesn't know agent 3. Agent 2 puts $1/2$ weight on his own opinion and $1/4$ on those of agents 1 and 3. Agent 3 puts $1/2$ weight on his own opinion and $1/2$ on the opinion of agent 2. He doesn't know agent 1. Thus, after initial discussion among the agents, their estimates have changed from $\mathbf{b}^{(0)}$ to

$$\mathbf{b}^{(1)} = \mathbf{T}\mathbf{b}^{(0)} = \begin{bmatrix} 7/8 \\ 1/2 \\ 1/4 \end{bmatrix}.$$

Note that

$$\mathbf{T}^2 = \begin{bmatrix} 5/8 & 5/16 & 1/16 \\ 5/16 & 7/16 & 1/4 \\ 1/8 & 1/2 & 3/8 \end{bmatrix},$$

so that after a second round of discussion, estimates have become

$$\mathbf{b}^{(2)} = \mathbf{T}^2 \mathbf{b}^{(0)} = \begin{bmatrix} 25/32 \\ 17/32 \\ 3/8 \end{bmatrix}.$$

In the limit,

$$\mathbf{T}^k \to \mathbf{T}^\infty = \begin{bmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{bmatrix}$$

so that no matter the initial estimates, and even though agents 1 and 3 don't know each other, all end up agreeing on the estimate as given by

$$\mathbf{b}^{(\infty)} = \lim_{k\to\infty} \mathbf{T}^k \mathbf{b}^{(0)} = \begin{bmatrix} 3/5 \\ 3/5 \\ 3/5 \end{bmatrix}.$$

That is, after "weighing the evidence", the consensus among agents is that the probability is $3/5$ that the official is guilty.

Under what conditions does the updating process just described converge to a well-defined limit? To characterize convergence, a few concepts from graph theory are useful. A *walk* in $\mathbf{T}$ is a sequence of nodes $i_1, i_2, \ldots, i_K$, not necessarily distinct, such that $T_{i_k i_{k+1}} > 0$ for each $k \in \{1, \ldots, K-1\}$. A *path* in $\mathbf{T}$ is a walk consisting of distinct nodes. A *cycle* is a walk $i_1, i_2, \ldots, i_K$ such that $i_1 = i_K$. A cycle is *simple* if the only node appearing twice in the sequence is the starting node.

Definition 1. A matrix $\mathbf{T}$ is *strongly connected* if there is a path in $\mathbf{T}$ from any node to any other node.

Definition 2. A matrix $\mathbf{T}$ is *aperiodic* if the greatest common divisor of the lengths of its simple cycles is 1.

Thus $\mathbf{T}$ is guaranteed to be aperiodic if $T_{ii} > 0$ for some i. Well-known results from Markov chain theory[3] on the necessity and sufficiency of aperiodicity for convergence is summarized in the following theorem from Golub and Jackson (2010).

Theorem 1. *(Golub and Jackson, 2010) If* $\mathbf{T}$ *is a strongly connected matrix, the following are equivalent:*

1. $\mathbf{T}$ *is convergent.*
2. $\mathbf{T}$ *is aperiodic.*
3. *There is a unique left eigenvector* π^T *of* $\mathbf{T}$ *corresponding to eigenvalue* 1 *whose entries sum to* 1 *such that, for every* $\mathbf{b}^{(0)} \in [0,1]^n$,

$$\left(\lim_{k\to\infty} \mathbf{T}^k \mathbf{b}^{(0)} \right)_i = \pi^\mathsf{T} \mathbf{b}^{(0)}$$

   for every i.

In addition to characterizing convergence, the theorem shows that when they converge, the limiting estimates are all equal to a weighted average of initial estimates, with agent i's weight being π_i. π_i is referred to as the *influence* of agent i with π being the vector of influences. For the example above, $\pi^\mathsf{T} = \begin{bmatrix} 2/5 & 2/5 & 1/5 \end{bmatrix}$. Thus agents 1 and 2 are twice as influential as agent 3.

To understand #3 of Theorem 1, suppose one wanted to find a nonnegative vector $\pi \in [0,1]^n$, the entries of which sum to 1, and which measures the influence of each agent on the consensus estimate. Since starting with $\mathbf{b}^{(0)}$ or $\mathbf{b}^{(1)}$ yields the same limit, it must be true that

$$\pi^\mathsf{T} \mathbf{b}^{(0)} = \pi^\mathsf{T} \mathbf{b}^{(1)} = \pi^\mathsf{T} \mathbf{T} \mathbf{b}^{(0)}$$

or

$$\pi^\mathsf{T} = \pi^\mathsf{T} \mathbf{T}.$$

Thus π is a left eigenvector of $\mathbf{T}$ with eigenvalue 1. The eigenvector property simply says that $\pi_j = \sum_i \pi_i T_{ij}$, i.e. the influence of agent j is a weighted sum of the influences of the agents i who listen to j, with the weight T_{ij} being the trust that agent i places in agent j's opinion. Said another way, one who is trusted by influential people is influential.

[3] See for example Kemeny and Snell (1960).

3. Differences of Opinion

It is desirable to have a way to measure differences of opinion along the way to consensus. A simple measure is the difference between the opinion of a single agent and the average opinion of all agents in the social network.[4] Let

$$\mathbf{e}_i = [\delta_{ij}] \text{ where } \delta_{ij} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \quad j = 1 \text{ to } n, \tag{4}$$

and

$$\mathbf{1} = \sum_{i=1}^{n} \mathbf{e}_i \tag{5}$$

i.e. $\mathbf{e}_i$ is the vector with 1 in the ith component, and 0's elsewhere and $\mathbf{1}$ is the vector with 1 in every component. Let $\mathbf{w}_{n\times 1}$ be an averaging vector such that for any $n \times 1$ vector $\mathbf{x}$

$$\mathbb{E}_{\mathbf{w}}[\mathbf{x}] = \mathbf{w}^{\mathsf{T}}\mathbf{x} \tag{6}$$

A natural choice for $\mathbf{w}$ is $1/n\mathbf{1}$, but $\mathbf{w}$ can be any weighted average including $\mathbf{e}_i$. Using $\mathbf{e}_i$ and $\mathbf{w}$, the difference between the opinion of agent i and the average opinion of all agents in the social network at update k is

$$\mathbf{e}_i^{\mathsf{T}}\mathbf{b}^{(k)} - \mathbf{w}^{\mathsf{T}}\mathbf{b}^{(k)} = \left(\mathbf{e}_i^{\mathsf{T}} - \mathbf{w}^{\mathsf{T}}\right)\mathbf{b}^{(k)} = \left(\mathbf{e}_i^{\mathsf{T}} - \mathbf{w}^{\mathsf{T}}\right)\mathbf{T}^k\mathbf{b}^{(0)}.$$

Let

$$\Delta_i^{(k)} = \left(\mathbf{e}_i^{\mathsf{T}} - \mathbf{w}^{\mathsf{T}}\right)\mathbf{T}^k$$

so that the difference between the opinion of agent i and the average opinion of all agents in the social network at update k is $\Delta_i^{(k)}\mathbf{b}^{(0)}$. Notice that $\Delta_i^{(k)}$ is the ith row of the matrix $\Delta^{(k)}$ where

$$\Delta^{(k)} = \left(\mathbf{I} - \mathbf{1}\mathbf{w}^{\mathsf{T}}\right)\mathbf{T}^k$$

$\mathbf{I}$ is the identity matrix. Thus $\Delta^{(k)}\mathbf{b}^{(0)}$ is the vector of differences for all agents. If $\lim_k \mathbf{T}^k$ exists, then so does $\lim_k \Delta^{(k)}$. These observations are summarized in the following definition.

Definition 3. Differencing Matrix

1. Let $\mathbf{T}$ be a social influence matrix and let $\mathbf{1}$ and $\mathbf{w}$ be as defined in Eqs.(5) and (6) respectively. Then

$$\Delta^{(k)} = \left(\mathbf{I} - \mathbf{1}\mathbf{w}^{\mathsf{T}}\right)\mathbf{T}^k \tag{7}$$

   is the *differencing matrix*[5] where $\Delta^{(k)}\mathbf{b}^{(0)}$ is the vector of differences between the opinions of each agent and the average (as defined by $\mathbf{w}$) opinion of all agents as of update k.
2. If $\lim_{k\to\infty}\mathbf{T}^k = \mathbf{T}^\infty$ then

$$\Delta^{(\infty)} = \lim_{k\to\infty}\Delta^{(k)} = \left(\mathbf{I} - \mathbf{1}\mathbf{w}^{\mathsf{T}}\right)\mathbf{T}^\infty$$

   and $\Delta^\infty\mathbf{b}^{(0)}$ is the final vector of differences where $\Delta^{(\infty)}\mathbf{b}^{(0)} = \left(\mathbf{I} - \mathbf{1}\mathbf{w}^{\mathsf{T}}\right)\mathbf{T}^\infty\mathbf{b}^{(0)} = \left(\mathbf{I} - \mathbf{1}\mathbf{w}^{\mathsf{T}}\right)\mathbf{b}^{(\infty)}$.

[4] DeMarzo et al. (2003).

[5] Although not explicitly derived by them, a version of the differencing matrix underlies the unidimensionality result proved by DeMarzo et al. (2003) in their Theorem 4.

As an example, using $\mathbf{w} = {}^1\!/_3\mathbf{1}$ with the matrix defined in Eq.(3) and initial estimates defined in Eq.(2),

$$\Delta^{(0)}\mathbf{b}^{(0)} = (\mathbf{I} - \mathbf{1}\mathbf{w}^{\mathsf{T}})\,\mathbf{T}^{0}\mathbf{b}^{(0)} = \begin{bmatrix} .5 \\ 0 \\ -.5 \end{bmatrix}$$

$$\Delta^{(2)}\mathbf{b}^{(0)} = (\mathbf{I} - \mathbf{1}\mathbf{w}^{\mathsf{T}})\,\mathbf{T}^{2}\mathbf{b}^{(0)} = \begin{bmatrix} .21875 \\ -.03125 \\ -.18750 \end{bmatrix}$$

$$\Delta^{(10)}\mathbf{b}^{(0)} = (\mathbf{I} - \mathbf{1}\mathbf{w}^{\mathsf{T}})\,\mathbf{T}^{10}\mathbf{b}^{(0)} = \begin{bmatrix} .00737 \\ -.00108 \\ -.00629 \end{bmatrix}$$

$$\Delta^{(\infty)}\mathbf{b}^{(0)} = (\mathbf{I} - \mathbf{1}\mathbf{w}^{\mathsf{T}})\,\mathbf{b}^{(\infty)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The differences vanish because, in the limit, all agents agree. This would be true for any $\mathbf{T}$ convergent under Proposition 1.

The differencing matrix will be used in Section 6.

4. Institutions, Convergence and Reducibility

An institution will be represented as an agent who directly influences the opinions of all other members of the social network but is not directly influenced by the opinion of any. That is, for agent i an institution, $T_{ii} = 1$, $T_{ij} = 0$ for all $j \neq i$ and $T_{ji} > 0$ for all j. Said another way, an institution is an agent whose opinion stays fixed at $b_i^{(0)}$ and to whom all other agents pay attention.[6] In reality, of course, an institution may not directly influence *every* member of society. Television news doesn't directly influence people who don't watch television. Likewise, decrees of the Catholic church don't directly influence non-Catholics. A similar caution applies to the requirement that institutional opinion not be influenced by any other agent. Nevertheless, as a first approximation, this representation captures institutional influence without unneccessary mathematical complications. In Section 7, the requirement that institutional opinion not be influenced by any other agent will be relaxed.

Consider the following social influence matrix where agent 3 is an institution.

$$\mathbf{T} = \begin{bmatrix} 2/5 & 1/5 & 2/5 \\ 1/10 & 2/5 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Notice that there is no path from agent 3 to agents 1 or 2 and so $\mathbf{T}$ is not strongly connected. Thus convergence is not guaranteed by Proposition 1. However, Perron-Frobenius theory applied to finite state Markov chains shows that, in fact, $\mathbf{T}$ *is* convergent. The demonstration that this is true requires some preliminary discussion. See Appendix B.

With all the preliminaries behind, the next theorem describes the conditions under which a reducible stochastic matrix is convergent.

Theorem 2. *Reducible Markov Chains*

Let $\mathbf{T}$ *be in canonical form*

$$\mathbf{T} = \begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \mathbf{0} & \mathbf{X}_{22} \end{bmatrix}$$

where

$$\mathbf{X}_{11} = \begin{bmatrix} \mathbf{T}_{11} & \cdots & \mathbf{T}_{1r} \\ & \ddots & \vdots \\ & & \mathbf{T}_{rr} \end{bmatrix}, \mathbf{X}_{12} = \begin{bmatrix} \mathbf{T}_{1,r+1} & \cdots & \mathbf{T}_{1m} \\ \vdots & & \vdots \\ \mathbf{T}_{r,r+1} & \cdots & \mathbf{T}_{rm} \end{bmatrix} \textit{ and } \mathbf{X}_{22} = \begin{bmatrix} \mathbf{T}_{r+1,r+1} & & \\ & \ddots & \\ & & \mathbf{T}_{mm} \end{bmatrix}$$

[6] See Section 8.3.1 in Jackson (2010).

and π_j be the left-hand Perron vector for $\mathbf{T}_{jj}$ $(r+1 \le j \le m)$. Then $\mathbf{I}-\mathbf{X}_{11}$ is nonsingular, and

$$\lim_{k\to\infty} \frac{\mathbf{I}+\mathbf{T}+\cdots+\mathbf{T}^{k-1}}{k} = \begin{bmatrix} \mathbf{0} & (\mathbf{I}-\mathbf{X}_{11})^{-1}\mathbf{X}_{12}\mathbf{E} \\ \mathbf{0} & \mathbf{E} \end{bmatrix}$$

where

$$\mathbf{E} = \begin{bmatrix} \mathbf{1}\pi_{r+1}^{\mathsf{T}} & & \\ & \ddots & \\ & & \mathbf{1}\pi_m^{\mathsf{T}} \end{bmatrix}.$$

Furthermore, $\lim_{k\to\infty}\mathbf{T}^k$ *exists if and only if the stochastic matrices* $\mathbf{T}_{r+1,r+1},\ldots,\mathbf{T}_{mm}$ *in* (B.1) *are each primitive, in which case*

$$\lim_{k\to\infty}\mathbf{T}^k = \begin{bmatrix} \mathbf{0} & (\mathbf{I}-\mathbf{X}_{11})^{-1}\mathbf{X}_{12}\mathbf{E} \\ \mathbf{0} & \mathbf{E} \end{bmatrix}. \tag{9}$$

5. Consensus: The Case of Joseph McCarthy

An example of an institution (the media) creating a consensus is provided by the political demise of Sen. Joseph McCarthy of Wisconsin,[7] a controversial politician of the 1950s. The media world of the 1950s was dominated by three television networks which captured more than 70% of Americans as a regular viewing audience. A healthy majority relied on network news, especially nightly news, as their primary source of information. The network news shows viewed themselves (and viewers agreed) as a public trust and were not required to be profit centers for the networks. Second in line were newspapers, the pages of which usually bent over backwards to report news objectively. In the 1950s Along with newspapers and newsreels, network news shows provided a common set of facts and a widely shared core of information.[8]

McCarthy had made his reputation by claims that there were large numbers of Communists and Soviet spies and sympathizers inside the U.S. federal government and U.S. military. In January of 1954, McCarthy was viewed favorably by 50% of the American public. By November, that figure had fallen to 35%. In the end, his tactics and inability to substantiate his claims ultimately led him to be censured by his colleagues in the United States Senate.

In popular culture, his downfall is often attributed to the March 1954 broadcast of an episode of the CBS documentary series “See It Now” entitled "A Report on Senator Joseph R. McCarthy."[9] The events leading up to that broadcast were re-created in the 2005 film, “Good Night and Good Luck.”[10]

Consider again the matrix in Eq.(8), reproduced here.

$$\mathbf{T} = \begin{bmatrix} 2/5 & 1/5 & 2/5 \\ 1/10 & 2/5 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

The social network consists of two agents and the media. Agent 1 puts equal weight on his own opinion and that of the media and only half as much on the opinion of the other agent. Agent 2 puts half of the weight on the media's opinion and four times more weight on his own opinion than the opinion of agent 1. Suppose that original opinions on some issue of interest are captured in $\mathbf{b}^{(0)} = [3,1,2]^{\mathsf{T}}$. Assuming the issue can be measured on a unidimensional scale, agents 1 and 2 are to the right and left of the media respectively. It is of interest to know if the members of the network, using the updating process of the DeGroot model, will reach a consensus and if so, what is the influence of the media on the consensus opinion. In this scenario, it will be seen that the media is an engine for societal consensus.

[7]See Wikipedia (2012). Joseph McCarthy. Wikimedia Foundation. URL: http://en.wikipedia.org/wiki/Joseph_McCarthy (visited on 06/27/2012).

[8]Mann and Ornstein (2012)

[9]The print media had turned against McCarthy much earlier. However, because of its national reach, the TV broadcast may have galvanized popular opinion. See Campbell (2010).

[10]*Good Night and Good Luck*. Dir. George Clooney. Warner Bros. Independent Films, 2005.

Notice that no path exists from the media to either of the agents so that $\mathbf{T}$ is reducible. Notice also that $\mathbf{T}$ is in canonical form as described in Appendix B. Finally, notice that for this simple example, $\mathbf{X}_{11} = \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & 2/5 \end{bmatrix}$, $\mathbf{X}_{12} = \begin{bmatrix} 2/5 \\ 1/2 \end{bmatrix}$ and $\mathbf{X}_{22} = [1]_{1\times 1} = \mathbf{E}$. Since $\mathbf{X}_{22}$ is (trivially) primitive, by Theorem 2,

$$\lim_{k\to\infty} \mathbf{T}^k = \begin{bmatrix} \mathbf{0} & (\mathbf{I}-\mathbf{X}_{11})^{-1}\mathbf{X}_{12} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

From Proposition 1, the influence vector is seen to be $\pi^{\mathsf{T}} = [0,0,1]$. A moment's reflection shows that this is a general result.

Proposition 1. *Let a social network consist of* $(n-1)$ *non-institutional agents and* 1 *institutional agent. Let the social influence matrix,*

$$\mathbf{T} = \begin{bmatrix} \mathbf{X} & \mathbf{y} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{11}$$

be in canonical form according to Eqs.(14) *and* (B.1) *where* $\mathbf{X}$ *is a nonnegative irreducible matrix with* $\rho = \rho(\mathbf{X}) < 1$ *and* $\mathbf{y}$ *is a vector. Assume all agents have an opinion concerning some topic of social interest and let agents' initial opinions be captured in the vector* $\mathbf{b}^{(0)}$. *Whether or not agents initially agree, in the end all agents will hold the institutional opinion. That is, if* $\mathbf{b}$ *is the vector of final opinions, then*

$$\mathbf{b} = b_n^{(0)}\mathbf{1}.$$

Proposition 1 shows that, in the DeGroot model with a strongly connected sub-network of non-institutional agents and a single institution, the institution's opinion becomes the consensus opinion. A proof is in Appendix A.

6. Polarization: The U.S. Congress Circa 2012

Consider the U.S. Congress circa 2012. There is general agreement that ideological polarization in the U. S. House and Senate had increased over the last generation, and that this change had contributed to Congressional deadlock in areas such as health care and deficit reduction. The following quote, taken from Mann and Ornstein (2012), a book by two congressional scholars, characterized the situation.

> Acrimony and hyperpartisanship have seeped into every part of the political process. Congress is deadlocked and its approval ratings are at record lows. America's two main political parties have given up their traditions of compromise, endangering our very system of constitutional democracy.

Mann and Ornstein went into detail.

> All the evidence on parties in government in recent years points to very high unity within and sharp ideological and policy differences between the two major parties. As *National Journal* reported in its study of roll call voting in the 111th Congress, for the first time in modern history, in both the House and Senate, the most conservative Democrat is slightly more liberal than the most liberal Republican. *This is another way of saying that the degree of overlap between the parties in Congress is zero.* (Italics added)

This Congressional schism can be represented within the DeGroot model by two institutional agents. If i is the index of the first institution and k the index for the second, $T_{kk} = 1$, $T_{kj} = 0$ for all $j \neq k$, $T_{jk} > 0$ for all $j \neq i$ and $T_{ik} = 0$. Of course, similar relations would hold for institutional agent i. As an example, let the institutions be liberal and conservative parties and consider

$$\mathbf{T} = \begin{bmatrix} 2/5 & 1/5 & 1/10 & 3/10 \\ 1/5 & 2/5 & 3/10 & 1/10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

Agent 1 considers the opinions expressed by the liberal party (agent 4) more valuable than those expressed by the conservative party (agent 3). In fact, he values liberal opinions three times as much as conservative opinions. Agent 2, on the other hand, is just the reverse. He values the opinions expressed by the conservative party three times more than those expressed by the liberal party. It is not surprising then that agent 2 values his own opinions twice as much as those of agent 1 nor that agent 1 feels likewise regarding the opinions of agent 2.

Let the original opinions of the agents be given by $\mathbf{b}^{(0)} = [2,3,4,1]^{\mathsf{T}}$. On a unidimensional scale, the parties represent the extremes. Agent 1, though left of center, is not as extreme as the liberal party. Likewise, agent 2 is right of center but not as extreme as the conservative party. As before, the questions of interest are the existence of consensus and the influence of the parties. It will be seen shortly that the presence of opposing institutions almost guarantees lack of consensus and social polarization.

Agents 3 and 4 can be seen as stand-ins for the Democratic and Republican parties in the U.S. circa 2012. Likewise, agents 1 and 2 can be seen as stand-ins for their liberal and conservative constituencies in the U.S. Congress. Although simple, the set-up is sufficiently realistic to reveal the connection between polarized (and polarizing) institutions and social polarization.

Having $\mathbf{e}_3^{\mathsf{T}}$ and $\mathbf{e}_4^{\mathsf{T}}$ as the final two rows on the right-hand side of (12) shows that $\mathbf{T}$ is reducible and that Theorem 2 applies. $\mathbf{X}_{11} = \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & 2/5 \end{bmatrix}$, $\mathbf{X}_{12} = \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 1/10 \end{bmatrix}$, $\mathbf{X}_{22} = \mathbf{X}_{33} = [1]_{1\times 1}$ and $\mathbf{E} = \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since $\mathbf{X}_{22}$ and $\mathbf{X}_{33}$ are primitive, by Theorem 2,

$$\lim_{k\to\infty} \mathbf{T}^k = \begin{bmatrix} \mathbf{0} & (\mathbf{I}-\mathbf{X}_{11})^{-1}\mathbf{X}_{12} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3/8 & 5/8 \\ 0 & 0 & 5/8 & 3/8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

Note that the lack of consensus is revealed in the presence of separate row vectors on the right side of Eq.(13) rather than a single influence vector as in Eq.(10). Note also that, as in Section 5, only the institutional opinions are given weight.

Finally,

$$\mathbf{b}^{(\infty)} = \lim_{k\to\infty} \mathbf{T}^k \mathbf{b}^{(0)} = \begin{bmatrix} 2\tfrac{1}{8} \\ 2\tfrac{7}{8} \\ 4 \\ 1 \end{bmatrix}.$$

Liberals have moved slightly right, conservatives have moved slightly left, but the network is still far from consensus.

The result (13) admits a minor generalization.

Corollary 1. *If* $\mathbf{X}_{22} = \mathbf{I}_{m-r}$ *then* $\mathbf{E} = \mathbf{I}_{m-r}$ *and*

$$\lim_{k\to\infty} \mathbf{T}^k = \begin{bmatrix} \mathbf{0} & (\mathbf{I}-\mathbf{X}_{11})^{-1}\mathbf{X}_{12} \\ \mathbf{0}_{m-r,n-m+r} & \mathbf{I}_{m-r} \end{bmatrix}$$

Represent $\mathbf{T}$, still in canonical form, as

$$\mathbf{T} = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{14}$$

and let

$$\lim_{k\to\infty} \mathbf{T}^k = \mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{Z} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

where

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix},$$

$$\mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$$

and

$$\mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = (\mathbf{I} - \mathbf{X})^{-1}\mathbf{Y}.$$

Explicitly calculating $\mathbf{Z}$,

$$\mathbf{Z} = \frac{1}{D}\begin{bmatrix} 1-x_{22} & x_{12} \\ x_{21} & 1-x_{11} \end{bmatrix}\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \tag{15}$$

$$= \frac{1}{D}\begin{bmatrix} (1-x_{22})y_{11} + x_{12}y_{21} & (1-x_{22})y_{12} + x_{12}y_{22} \\ x_{21}y_{11} + (1-x_{11})y_{21} & x_{21}y_{12} + (1-x_{11})y_{22} \end{bmatrix} \tag{16}$$

where D, the determinant, is

$$D = (1-x_{11})(1-x_{22}) - x_{12}x_{21}. \tag{17}$$

Now

$$\mathbf{b}^{(\infty)} = \mathbf{G}\mathbf{b}^{(0)} = \begin{bmatrix} z_{11}b_3^{(0)} + z_{12}b_4^{(0)} \\ z_{21}b_3^{(0)} + z_{22}b_4^{(0)} \\ b_3^{(0)} \\ b_4^{(0)} \end{bmatrix}.$$

It is clear that the political party agents will never agree, but it is still possible to claim a *partial consensus* if agents 1 and 2 are able to agree. To examine the extent of disagreement between agents 1 and 2, let $\mathbf{w} = \mathbf{e_1}$ and apply the differencing matrix from Section 3.

$$\Delta^{(\infty)}\mathbf{b}^{(0)} = \begin{bmatrix} 0 \\ (z_{21} - z_{11})b_3^{(0)} + (z_{22} - z_{12})b_4^{(0)} \\ (1 - z_{11})b_3^{(0)} - z_{12}b_4^{(0)} \\ -z_{11}b_3^{(0)} + (1 - z_{12})b_4^{(0)} \end{bmatrix}$$

The first component is 0 because agent 1 agrees with himself. The second component shows that agents 1 and 2 will agree if

$$(z_{21} - z_{11})b_3^{(0)} + (z_{22} - z_{12})b_4^{(0)} = 0 \tag{18}$$

Looking at the matrix on the right-hand side of (16), the initial thought might be to set $(1 - x_{11}) = x_{12}$ and $(1 - x_{22}) = x_{21}$. However, a quick check of Eq.(17) shows that would make $\mathbf{I} - \mathbf{X}$ singular. Another way to satisfy condition (18) is to set $y_{ij} = y$ for all i and j. The condition that row sums equal 1 would then require that

$$x_{11} + x_{12} = x_{21} + x_{22} = 1 - c \tag{19}$$

and

$$y = \frac{c}{2}.$$

Then, from (16),

$$\mathbf{Z} = \frac{c}{2D}\begin{bmatrix} 1 + x_{12} - x_{22} & 1 + x_{12} - x_{22} \\ 1 + x_{21} - x_{11} & 1 + x_{21} - x_{11} \end{bmatrix} = \frac{c(1+a)}{2D}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

where, from (19), $a = x_{12} - x_{22} = x_{21} - x_{11}$. Thus society can achieve consensus if the members agree on their valuations of institutional opinions.

As an example, change the $\mathbf{T}$ matrix in Eq.(12) to

$$\hat{\mathbf{T}} = \begin{bmatrix} \mathbf{X} & \hat{\mathbf{Y}} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 2/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 2/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $D = 8/25$, $a = -1/5$, $c = 2/5$ and

$$\hat{\mathbf{Z}} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\hat{\mathbf{b}} = \hat{\mathbf{G}}\mathbf{b}^{(0)} = \begin{bmatrix} \mathbf{0} & \hat{\mathbf{Z}} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\mathbf{b}^{(0)} = \begin{bmatrix} 5/2 \\ 5/2 \\ 4 \\ 1 \end{bmatrix}$$

and agents 1 and 2 are seen to be in agreement.

In this example, the opinions of both institutions were valued equally but they need not be. All that is required is that the agents *agree* on the valuation. This result generalizes to many agents and many institutions. The following proposition, proved in Appendix A, sets out the conditions.

Proposition 2. *Let a network consist of* $(n-k)$ *non-institutional agents and k institutional agents. Let the social influence matrix,*

$$\mathbf{T} = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$

be in canonical form according to Eqs.(14) *and* (B.1) *so that* $\mathbf{X}$ *is a nonnegative irreducible matrix with* $\rho = \rho(\mathbf{X}) < 1$. *Let some institutional agents disagree about a topic of social interest and let some non-institutional agents initially disagree as well. Let the vector of initial opinions on the topic be* $\mathbf{b}^{(0)}$ *where*

$$b_i^{(0)} \neq b_j^{(0)} \text{ for some } i, j > n-k$$

and

$$b_i^{(0)} \neq b_j^{(0)} \text{ for some } i, j \leq n-k.$$

If partial consensus is defined as agreement between non-institutional agents, then the society reaches a partial consensus if and only if

$$\mathbf{Y} = \mathbf{1}\mathbf{v}^{\mathsf{T}} \tag{20}$$

for some vector $\mathbf{v}$.

Proposition 2 says that partial consensus will be reached only if all non-institutional agents agree on the valuation of institutional opinions. Casual empiricism regarding human affiliations and ensuing loyalties suggests that such agreement, though not impossible, is highly unlikely.

7. Institutional Insularity and Time to Consensus: The Conflict in Northern Ireland

Institutions transcend individual human lives. Consider the conflict in Northern Ireland. It’s beginnings can be credibly dated to Henry VIII’s break with the Catholic Church in Rome. Under Henry VIII, The Act of Supremacy in 1534 declared that the King was "the only Supreme Head in Earth of the Church of England." In the same year, an Irish revolt by Thomas, Lord Offaly, was swiftly put down by Henry. Significantly, Offaly had attempted to rally the Irish in the cause of a ’Catholic crusade’ against the Protestant English king, introducing religion into Irish politics for the first time.[11] These events set the stage for the violence between Protestants and Catholics in Ireland which continued through the end of the twentieth century, nominally ending with the Belfast "Good Friday" Agreement in 1998, although sporadic violence has continued since then.[12] Thus it has taken approximately five centuries - 500 years or 25 generations - for Northern Irish Protestants and Catholics to (hopefully) come to a consensus on the conditions under which the two sides can live together in peace.

In trying to uncover the conditions that led to this conflict lasting so long, note that the time to consensus depends on the time to convergence of a society's social influence matrix, $\mathbf{T}$. It turns out that the convergence of $\mathbf{T}^k$ to the limit $\mathbf{T}^\infty$ is governed by how quickly λ_2^k goes to 0 where λ_2 is the second largest eigenvalue of $\mathbf{T}$.[13] This bears on the question of interest - the impact of institutions on the time to consensus. Thus far, the analysis has assumed that the institutions are unchanging, unaffected by other agents in the society. This is, of course, an idealization. In reality, institutions are run by human beings. Humans die and are replaced by others. Replacements bring the ideas and attitudes of the era in which they live into the institution and so, institutions do change.

To explore this, consider a society with two opposing religious institutions. Both institutions claim that they listen only to God, and so imagine that the social influence matrix is $\mathbf{T}$ as given in Eq.(12). But because, as mentioned above, human life spans are finite and because institutions reflect their times, even if only indirectly, the actual matrix is $\widetilde{\mathbf{T}}$ where

$$\widetilde{\mathbf{T}} = \begin{bmatrix} 2/5 & 1/5 & 1/10 & 3/10 \\ 1/5 & 2/5 & 3/10 & 1/10 \\ \varepsilon/2 & \varepsilon/2 & 1-\varepsilon & 0 \\ \varepsilon/2 & \varepsilon/2 & 0 & 1-\varepsilon \end{bmatrix}. \tag{21}$$

The institutions are not, as they believe, completely isolated from society because $T_{ii} < 1, i = 3,4$. However, rather than being the weight a religious institution puts on secular opinion,[14] $\varepsilon/2$ is more a measure of the influence, unintended by the institution, that a secular agent has on a religious institution. The right-hand side of (21) reflects the assumption that this is small and is the same for all secular agents. The zeroes reflect the assumption that religious institutions cannot directly influence each other because of religious animosity. However, as can be seen from (21), they do *indirectly* influence each other. For agent i a member of the society with social influence matrix $\mathbf{T}$, let the measure of the *insularity* of agent i be T_{ii}.[15] As seen in Sections 5 and 6, when insularity is 1 agent i is not influenced by any other agent in the society. The insularity of the institutions in $\widetilde{\mathbf{T}}$ however, is not 1 but $1-\varepsilon$.

$\widetilde{\mathbf{T}}$ is an example of an almost reducible (some authors say "nearly completely decomposable") stochastic matrix. The states naturally divide into k clusters such that the states within each cluster are strongly coupled, but the clusters themselves are only weakly coupled to each other. A result developed by Hartfiel and Meyer (1998) can be used to relate the second largest eigenvalue of such a matrix to institutional insularity. Let $\mathbf{T}_{n\times n}$ be a stochastic matrix. These authors define the *coupling measure* of $\mathbf{T}$ as

$$\mu(\mathbf{T}) = \min \left(\sum_{\substack{i\in M_1 \\ j\notin M_1}} T_{ij} + \sum_{\substack{i\in M_2 \\ j\notin M_2}} T_{ij} \right) \tag{22}$$

where the minimum is taken over all nonempty proper subsets M_1, M_2 of $\{1,\ldots,n\}$ with $M_1 \cap M_2 = \emptyset$. Suppose $\mathbf{T}$ has been rearranged so that $\mu(\mathbf{T})$ is achieved at $M_1 = \{k_1+1, k_1+2, \ldots, k_2\}$, $M_2 = \{k_2+1, k_2+2, \ldots, n\}$ then $\mathbf{T}$ can

[11] BBC History (2009). The Road to Northern Ireland, 1167 to 1921 (2007). British Broadcasting Corporation. URL: http://www.bbc.co.uk/history/recent/troubles/overview_ni_article_01.shtml (visited on 06/26/2012).

[12] See Wikipedia (2012). The Troubles. Wikimedia Foundation. URL: http://en.wikipedia.org/wiki/Northern_Ireland_conflict (visited on 06/26/2012).

[13] See Section 8.3.6 in Jackson (2010) for an intuitive explanation. Also cf. Section 7.2 in Meyer (2000).

[14] Because of the religious institutions' refusal to listen to secular voices, the weight is zero.

[15] DeMarzo et al. (2003) label T_{ii} "self-importance."

be partitioned as

$$\mathbf{T} = \begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} & \mathbf{X}_{13} \\ \mathbf{X}_{21} & \mathbf{X}_{22} & \mathbf{X}_{23} \\ \mathbf{X}_{31} & \mathbf{X}_{32} & \mathbf{X}_{33} \end{bmatrix} \tag{23}$$

where $\mathbf{X}_{11}$ is $k_1 \times k_1$, $\mathbf{X}_{22}$ is $(k_2 - k_1) \times (k_2 - k_1)$ and $\mathbf{X}_{33}$ is $(n - k_2) \times (n - k_2)$. Then $\mu(\mathbf{T})$ is the sum of the entries in $\mathbf{X}_{21}$, $\mathbf{X}_{23}$, $\mathbf{X}_{31}$, and $\mathbf{X}_{32}$. If $\mu(\mathbf{T})$ is small, $\mathbf{T}$ is nearly uncoupled into 3 blocks - $\mathbf{X}_{11}$, $\mathbf{X}_{22}$ and $\mathbf{X}_{33}$.[16] If $\mathbf{T}$ is structured so that the last m agents are institutions, then

$$\mu(\mathbf{T}) = \min_{i \neq j} (2 - T_{ii} - T_{jj}) \quad i, j \in \{k_1 + 1, \ldots, k_1 + m\}$$

Hartfiel and Meyer (1998) then prove the following relationship between the second largest eigenvalue $\lambda_2(\mathbf{T})$ and the coupling measure $\mu(\mathbf{T})$.

Theorem 3. *(Hartfiel and Meyer, 1998) For any $\varepsilon > 0$ there is a $\delta > 0$ such that if $\mathbf{T}_{n\times n}$ is a strongly connected stochastic matrix with $\mu(\mathbf{T}) < \delta$, then $|\lambda_2(\mathbf{T})| > 1 - \varepsilon$. Conversely, for any $\delta > 0$ there is an $\varepsilon > 0$ such that if $\mathbf{T}_{n\times n}$ is a strongly connected stochastic matrix with $|\lambda_2(\mathbf{T})| > 1 - \varepsilon$, then $\mu(\mathbf{T}) < \delta$.*

The idea is that if the coupling measure is low, there are two disjoint groups that pay little attention to anyone outside of their respective groups. In such a situation, these groups can maintain separate beliefs for a long time, convergence is relatively slow and time to consensus protracted. Note that if there is only one group that is introspective, then the rest of society must be paying attention to that group and convergence can still be fast.[17] The coupling measure makes clear that there must be at least *two* groups with little communication.

It was shown in Section 6 that a society with social influence matrix $\mathbf{T}$ in (12) would never achieve full consensus. Clearly, for such a society, $\mu(\mathbf{T}) = 0$. By contrast, $\mu\left(\widetilde{\mathbf{T}}\right) = 2\varepsilon$. In addition, $\widetilde{\mathbf{T}}$ is strongly connected and so is convergent (by Theorem 1). Thus, unlike Section 6, the society described by $\widetilde{\mathbf{T}}$ *will* reach a consensus. But how long will consensus take?

The time to consensus has been studied by Golub and Jackson (2008). These authors define time to consensus - *CT* - as follows.

Definition 4. (Golub and Jackson, 2008)The consensus time to $\gamma > 0$ of a network with social influence matrix $\mathbf{T}$ is

$$CT(\gamma) = \sup_{\mathbf{b}\in[0,1]^n} \min\left\{k : \left\|\mathbf{T}^k\mathbf{b} - \mathbf{T}^\infty\mathbf{b}\right\|^2 < \gamma\right\}$$

where $\mathbf{T}^\infty = \lim_{k\to\infty} \mathbf{T}^k$ and $\mathbf{b} = \lim_{k\to\infty} \mathbf{T}^k\mathbf{b}^{(0)}$.[18] They then show that if $\mathbf{T}$ is strongly connected, *CT* is bounded.

Theorem 4. *(Golub and Jackson, 2008)[19] Assume $\mathbf{T}$ is strongly connected, let λ_2 be the second largest eigenvalue in magnitude of $\mathbf{T}$ and let π be the (unique) steady-state distribution, with $\min_i \pi_i = \underline{\pi}$. If $\lambda_2 \neq 0$, then for any $0 < \gamma \leq 1$:*

$$\left\lfloor \frac{1}{2} \frac{\log\left(\frac{4\gamma}{\underline{\pi}}\right)}{\log(\lambda_2)} \right\rfloor \leq CT(\gamma) \leq \left\lceil \frac{1}{2} \frac{\log(\gamma)}{\log(\lambda_2)} \right\rceil. \tag{24}$$

The functions $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are *floor* and *ceiling* respectively.

Two items related to Theorem 4 deserve mention. First, as noted above, a key insight is that the convergence time of an iterated stochastic matrix is related to its second largest eigenvalue. Second, Theorems 3 and 24 together show how coupling affects time to consensus. Although it is, in general, not possible to directly map coupling to

[16] This definition can, of course, be extended to more than than 3 blocks.

[17] Recall the case of a single institution analyzed in Section 5.

[18] cf. Theorem 1.

[19] Lemma 3.

μ	$-\frac{1}{\log\left(1-\frac{\mu}{2}\right)}$	**LB**	**UB**
2×10^{-1}	9.5	1	15
2×10^{-2}	99.5	11	264
2×10^{-3}	999.5	111	3,799
2×10^{-4}	9,999.5	1,115	49,515
2×10^{-5}	99,999.5	11,157	610,301
2×10^{-6}	999,999.5	111,571	7,254,326

Table 1: LB and UB are the bounds, lower and upper, on time to convergence within $\varepsilon/2$ of consensus

consensus, $\widetilde{\mathbf{T}}$ can be used to demonstrate a special case. Recall that eigenvalues of a square matrix are solutions to its *characteristic equation.* The characteristic equation of $\widetilde{\mathbf{T}}$ is

$$\frac{1}{25}\left(\lambda-1\right)\left(\lambda-[1-\varepsilon]\right)\left(5\lambda-1\right)\left(5\lambda-[3-5\varepsilon]\right)=0$$

It's easy to see that for small values of ε, the second largest eigenvalue is $\lambda_2=1-\varepsilon$. Also, the steady-state distribution is the left eigenvector associated with the spectral radius $\rho=1$ of $\widetilde{\mathbf{T}}$ and is found to be $\mathbf{s}=\left[{}^{5\varepsilon}\!/\!{}_2,{}^{5\varepsilon}\!/\!{}_2,1,1\right]^{\mathsf{T}}$. Thus the bounds can be expressed as functions of ε and γ. To keep things simple, let $\gamma={}^{\varepsilon}\!/\!{}_2$ and recall that $\mu(\mathbf{T})=2\varepsilon$. Then

$$\left\lfloor\frac{1}{2}\frac{\log\left(\frac{4}{5}\right)}{\log\left(1-\frac{\mu}{2}\right)}\right\rfloor\leq CT\left(\mu\right)\leq\left\lceil\frac{1}{2}\frac{\log\left(\frac{\mu}{4}\right)}{\log\left(1-\frac{\mu}{2}\right)}\right\rceil \tag{25}$$

CT will be the time required for the society to come within ${}^{\mu}\!/\!{}_4$ (${}^{\varepsilon}\!/\!{}_2$) of complete consensus. The results for several values of μ (2ε) are shown in Table 1. The table shows that when the influence of any secular agent on a religious institution is really small - 10^{-5}, say - the time to convergence can be really long, involving over 610 thousand rounds of updating. This many updates could constitute time scales covering many generations of agents.

Suppose $\widetilde{\mathbf{T}}$ is the social influence matrix for the two sides and that the Irish update opinions once a day. One can get a back-of-the-envelope estimate of the coupling μ and so get a measure of the insularity of the institutions vis-a-vis the general population. The lower the value of μ, the more insular the institutions. Using the upper-bound inequality in (25) and assuming 365 days in a year, the coupling estimate is $\mu=6.1\times10^{-5}$ implying ${}^{\varepsilon}\!/\!{}_2=1.5\times10^{-5}$. Thus, if $\widetilde{\mathbf{T}}$ were the actual social influence matrix for Northern Ireland, there was only a .0015% chance that the institutions fighting the war could be influenced by the people in whose name they fought.[20]

In addition, the coupling estimate allows an estimation of the lower bound, resulting in 40.2 years $\leq CT\leq$ 500 years. In other words, given such a small value for institutional coupling, in the *best* case, the war would have lasted at least 40 years. Of course, the numbers used for the calculation are "made-up." Nevertheless, given the length of the actual conflict - 500 years - the impediment to societal consensus represented by insularity of opposing institutions is sobering.

8. Conclusion

Two fundamental questions in social network theory are 1) how social institutions influence whether or not the opinions of individuals converge to a common societal opinion and 2) if individual opinions do converge, how that common opinion is influenced by the presence of social institutions. To examine these questions, I represent social institutions within the framework of the well-known DeGroot model of information transmission and consensus formation. Citing standard results in matrix analysis and linear algebra, I show that the representation within the DeGroot

[20] A more current example might be ISIS, an organization that engages in mass killing in the name of Islam, apparently impervious to the war-weariness of the Islamic populations upon whom they inflict their depredations.

framework is equivalent to a reducible Markov chain. Using convergence results for such chains, I show that in the case of a single institution, the institutional opinion becomes the consensus opinion.

Using the same representation for institutions, I consider the case of two opposing institutions. Because the institutions will never agree societal consensus is impossible. I define partial consensus as consensus among the non-institutional agents in the network. I formulate a way of measuring disagreement among agents in a social network - the differencing matrix - and use it to derive the conditions under which a partial rather than a societal consensus can be reached. I show that a partial consensus can be achieved only if all non-institutional agents agree on the valuation of institutional opinions. I argue that this condition, though theoretically possible, is highly unlikely. For example among U.S. political parties, one would expect that Democrats would value Democratic more than Republican opinions and vice-versa. Thus I conclude that in the case of opposing institutions, social polarization is virtually assured.

Still another question in social network theory is if society does reach a consensus, how long does it take. To discuss this, I introduce the notion of insularity, i.e. disconnectedness from the network, and relax the requirement that institutional insularity equal 1. I cite a standard result from linear algebra that the speed of convergence of a strongly connected stochastic matrix is governed by the size of its second largest eigenvalue. Hartfiel and Meyer (1998) define a quantity - the coupling measure - that measures the "combined disconnectedness" of two clusters of states in an irreducible Markov chain. They use it to prove a theorem that connects the size of the coupling measure to the size of the second largest eigenvalue. Golub and Jackson (2008) prove a theorem that shows how bounds on the time to consensus of a strongly connected stochastic matrix depend on the size of the second largest eigenvalue. I use the two theorems to discuss how institutional insularity affects time to consensus.

The analysis in this paper is focused on the influence that institutions have on societal consensus. In the analysis I assume that social networks consist of institutional and non-institutional agents. As a simplification, I assume that the non-institutional agents are strongly connected. That means that I am assuming that every non-institutional agent is influenced by every other non-institutional agent because, even if two agents don't communicate with each other directly, they are connected by a sequence of agents who do communicate directly with each other. The idea is captured in the well-known phrase "six degrees of separation" made popular by the play (Guare, 1990) of the same name. This is, of course, not true in general. When there are several non-communicating strongly connected groups, each reaches its own consensus, with its own social influence weights, and then the remaining agents who are path-connected to the strongly connected groups acquire some weighted average of the beliefs of the strongly connected groups. This can be seen in the analysis of Section 6 where the non-communicating strongly connected groups are, in fact, the institutions. The case of such groups embedded among the non-institutional agents has been adequately covered in the literature, e.g. (Golub and Jackson, 2010). Analyzing it here would add complications with no additional insight. Similar remarks apply to the case of more than two institutions.

References

Bonacich, P., 1987. Power and Centrality: A Family of Measures. American Journal of Sociology 92, 1170–1182.

Campbell, W.J., 2010. Media ?too scared? to challenge Joe Mccarthy? Hardly. URL: URLhttp://mediamythalert.wordpress.com/2010/07/14/media-too-scared-to-challenge-joe-mccarthy-hardly/.

Chatterjee, S., Seneta, E., 1977. Towards consensus: Some convergence theorems on repeated averaging. Journal of Applied Probability 14, 89–97.

DeGroot, M.H., 1974. Reaching a consensus. Journal of the American Statistical Association 69, 118–121.

DeMarzo, P., Vayanos, D., Zwiebel, J., 2003. Persuasion Bias, Social Influence, and Unidimensional Opinions. The Quarterly Journal of Economics 118, 909–968.

French Jr., J., 1956. A Formal Theory of Social Power. Psychological Review 63, 181.

Friedkin, N., Johnsen, E., 1990. Social influence and opinions. Journal of Mathematical Sociology 15, 193–206.

Golub, B., Jackson, M., 2008. How Homophily Affects Diffusion and Learning in Networks. arXiv preprint arXiv:0811.4013 .

Golub, B., Jackson, M., 2010. Naive Learning in Social Networks and the Wisdom of Crowds. American Economic Journal: Microeconomics 2, 112–49.

Guare, J., 1990. Six Degrees of Separation: A Play. Vintage.

Harary, F., 1959. A criterion for Unanimity in French's Theory of Social Power .

Hartfiel, D., Meyer, C., 1998. On the Structure of Stochastic Matrices with a Subdominant Eigenvalue Near 1. Linear Algebra and Its Applications 272, 193–203.

Jackson, M., 2010. Social and Economic Networks. Princeton University Press.

Katz, L., 1953. A New Status Index Derived from Sociometric Analysis. Psychometrika 18, 39–43.

Kemeny, J., Snell, J., 1960. Finite Markov Chains. Van Nostrand.

Krause, U., 2000. A Discrete Nonlinear and Non-Autonomous Model of Consensus Formation. Communications in difference equations 2000, 227–236.

Mann, T., Ornstein, N., 2012. It's Even Worse Than It Looks: How the American Constitutional System Collided with the New Politics of Extremism. Basic Books.

Meyer, C., 2000. Matrix Analysis and Applied Linear Algebra. volume 71. Society for Industrial and Applied Mathematics.

Appendix A. Proofs

Appendix A.1. Proof of Proposition 1

Proof. Using Eq.(11) in Theorem 2, note that $\mathbf{E} = [1]_{1\times 1}$ and, because the row sums of $\mathbf{y}$ equal the row sums of $(\mathbf{I}-\mathbf{X})$, $\mathbf{y} = (\mathbf{I}-\mathbf{X})\,\mathbf{1}$. Substitution into Eq.(9) gives

$$\lim_{k\to\infty} \mathbf{T}^k = \mathbf{G} = \begin{bmatrix} \mathbf{0}_{n-1,n-1} & \mathbf{1}_{n-1,1} \\ \mathbf{0}_{1,n-1} & 1_{1\times 1} \end{bmatrix}$$

and $\mathbf{b} = \mathbf{G}\mathbf{b}^{(0)} = b_n^{(0)}\mathbf{1}$. □

Appendix A.2. Proof of Proposition 2

Theorem 5. *Meyer (2000)*[21] *Let* $\mathbf{A}_{n\times n}$ *be a real matrix with* $\rho(\mathbf{A}) < 1$. *Then* $(\mathbf{I}-\mathbf{A})^{-1}$ *exists and*

$$(\mathbf{I}-\mathbf{A})^{-1} = \mathbf{I} + \sum_{k=1}^{\infty} \mathbf{A}^k. \tag{A.1}$$

Lemma 1. *If* $\mathbf{1}$ *is an eigenvector of* $\mathbf{A}$*, then* $\mathbf{1}$ *is an eigenvector of* $\mathbf{A}^k$.

Proof. Let $\mathbf{A}\mathbf{1} = \alpha\mathbf{1}$. Then $\mathbf{A}^k\mathbf{1} = \mathbf{A}^{k-1}\mathbf{A}\mathbf{1} = \alpha\mathbf{A}^{k-1}\mathbf{1} = \alpha^2\mathbf{A}^{k-2}\mathbf{1} = \ldots\alpha^k\mathbf{1}$. □

Lemma 2. *Let* $\mathbf{A}_{n\times n}$ *be a nonnegative irreducible matrix with* $\rho = \rho(\mathbf{A}) < 1$. *If* $\mathbf{1}$ *is an eigenvector of* $\mathbf{A}$*, then* $\mathbf{1}$ *is an eigenvector of* $(\mathbf{I}-\mathbf{A})^{-1}$ *with eigenvalue* $1/(1-\rho)$.

Proof. By Theorem 5, $(\mathbf{I}-\mathbf{A})^{-1}$ exists. By Theorem 7, $\mathbf{A}\mathbf{1} = \rho\mathbf{1}$. Then

$$(\mathbf{I}-\mathbf{A})^{-1}\mathbf{1} = \left(\mathbf{I} + \sum_{k=1}^{\infty} \mathbf{A}^k\right)\mathbf{1} = \left(1 + \sum_{k=1}^{\infty} \rho^k\right)\mathbf{1} = \frac{1}{1-\rho}\mathbf{1}$$

□

Proposition 2

Proof. Note that the dimensions of $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{1}$ are $(n-k)\times(n-k)$, $(n-k)\times k$ and $(n-k)$respectively. The dimensions of $\mathbf{I}$ are $(n-k)\times(n-k)$.

Let $\beta = \begin{bmatrix} b_{n-k+1}^{(0)} \\ \vdots \\ b_n^{(0)} \end{bmatrix}$ and $\mathbf{Y} = \mathbf{1}\mathbf{v}^\intercal$, i.e. Eq.(20) holds. To prove sufficiency, observe that if a consensus is to be reached, then from Theorem 2,

$$(\mathbf{I}-\mathbf{X})^{-1}\mathbf{Y}\beta = z\mathbf{1} \tag{A.2}$$

for some z. Now

$$(\mathbf{I}-\mathbf{X})^{-\mathbf{1}}\mathbf{1}\mathbf{v}^\intercal\beta = (\mathbf{I}-\mathbf{X})^{-1}y\mathbf{1}$$

where $\mathbf{v}^\intercal\beta = y$. From the definition of $\mathbf{Y}$, the row sums of $\mathbf{Y}$ are all equal. Because corresponding rows of $\mathbf{X}$ and $\mathbf{Y}$ must sum to 1, this means that the row sums of $\mathbf{X}$ are also all equal. Thus, by Lemma 2, $\mathbf{1}$ is an eigenvalue of $(\mathbf{I}-\mathbf{X})^{-\mathbf{1}}$, $z = y/(1-\rho)$ and agents reach a consensus. The vector of final opinions is $\mathbf{b} = \begin{bmatrix} y/(1-\rho)\mathbf{1} \\ \beta \end{bmatrix}$.

[21] p. 618

To prove necessity, assume that Eq.(A.2) holds but that Eq.(20) does not. Since Eq.(A.2) holds,

$$\mathbf{Y}\beta = (\mathbf{I} - \mathbf{X})z\mathbf{1}.$$

Because the row sums of $\mathbf{Y}$ are unequal, so are the row sums of $\mathbf{X}$. This means that $\mathbf{1}$ is now not an eigenvector of $\mathbf{X}$. Thus,

$$\mathbf{Y}\beta = z(\mathbf{I} - \mathbf{X})\mathbf{1} = z\,(\mathbf{1} - \mathbf{X}\mathbf{1})\,.$$

Since $\mathbf{X1}$ is the vector of row sums of $\mathbf{X}$, $(\mathbf{1} - \mathbf{X1}) = \mathbf{Y1}$ is the vector of row sums of $\mathbf{Y}$ and so

$$\mathbf{Y}\beta = \mathbf{Y}\,(z\mathbf{1})\,.$$

This equality can only be true if $\beta_1 = \beta_2 = \ldots = \beta_k$. However, that contradicts the assumption that $b_i^{(0)} \neq b_j^{(0)}$ for some i and j greater than $n - k$, i.e. the assumption that at least two institutions disagree. The contradiction shows that (A.2) holds only if Eq.(20) does. □

Appendix B. Convergence and Reducibility

The set of distinct eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, $k \leq n$ for the square matrix $\mathbf{T}_{n\times n}$ is called the *spectrum* of $\mathbf{T}$ and is denoted $\sigma(\mathbf{T})$. If m of the n eigenvalues of $\mathbf{T}$ equal λ, then λ is said to have *algebraic multiplicity m*.

Definition 5. For a square matrix $\mathbf{T}$, the number

$$\rho\,(\mathbf{T}) = \max_{\lambda \in \sigma(\mathbf{T})} |\lambda|$$

is called the *spectral radius* of $\mathbf{T}$.

Theorem 6. *For a stochastic matrix* $\mathbf{T}$*,* $\rho\,(\mathbf{T}) = 1$.

For a stochastic matrix $\mathbf{T}$, if $\rho\,(\mathbf{T})$ has multiplicity m and the n eigenvalues of $\mathbf{T}$ are ranked in order of descending absolute value, then $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_m| = 1$, and $\left|\lambda_j\right| < 1 \quad j = m+1$ to n.

Every stochastic matrix defines a Markov chain and vice-versa. A strongly connected stochastic matrix is an *irreducible* Markov chain.

Theorem 7. *Perron-Frobenius Theorem*
If $\mathbf{T}_{n\times n} \geq 0$ *is irreducible, then the following is true.*

1. $\rho\,(\mathbf{T}) \in \sigma(\mathbf{T})$ *and* $\rho\,(\mathbf{T}) = \rho > 0$.
2. *The algebraic multiplicity of* $\rho\,(\mathbf{T})$ *is* 1.
3. *There exists an eigenvector* $\mathbf{x} > 0$ *such that* $\mathbf{Tx} = \rho\mathbf{x}$.
4. *The unique vector defined by* $\mathbf{T}\pi = \rho\pi,\ \pi > 0$ *and* $\sum_j \pi_j = 1$ *is called the* Perron vector. *There are no nonnegative eigenvectors for* $\mathbf{T}$ *except for positive multiples of* π*, regardless of the eigenvalue.*

A stochastic matrix that is not strongly connected is *reducible*. Because the Perron-Frobenius theorem is not directly applicable to reducible chains, the strategy is to express reducible chains in terms of irreducible chains.

Definition 6. Reducibility

1. If $\mathbf{H}$ is a permutation matrix[22], $\mathbf{H}^\intercal\mathbf{T}\mathbf{H}$ is called a *symmetric permutation* of $\mathbf{T}$. The effect is to interchange rows in the same way as columns are interchanged.

[22] See, for example, chap. 3 in Meyer (2000) for a definition and discussion.

2. $\mathbf{T}_{n\times n}$ is said to be *reducible* when there exists a permutation matrix $\mathbf{H}$ such that

$$\mathbf{H}^\mathsf{T}\mathbf{T}\mathbf{H} = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{bmatrix}$$

where $\mathbf{X}$ and $\mathbf{Z}$ are both square. Otherwise $\mathbf{T}$ is said to be *irreducible*.

If $\mathbf{T}$ is reducible, by definition the rows and columns of $\mathbf{T}$ can be rearranged as in Definition 6.2. Denote this by

$$\mathbf{T} \sim \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{bmatrix}$$

If $\mathbf{X}$ or $\mathbf{Z}$ is reducible, another symmetric permutation can be performed to produce

$$\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{bmatrix} \sim \begin{bmatrix} \mathbf{R} & \mathbf{S} & \mathbf{T} \\ \mathbf{0} & \mathbf{U} & \mathbf{V} \\ \mathbf{0} & \mathbf{0} & \mathbf{W} \end{bmatrix}$$

where $\mathbf{R}$, $\mathbf{U}$ and $\mathbf{W}$ are square. Repeating this process eventually yields

$$\mathbf{T} \sim \begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} & \cdots & \mathbf{X}_{1k} \\ \mathbf{0} & \mathbf{X}_{22} & \cdots & \mathbf{X}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{X}_{kk} \end{bmatrix}$$

where each $\mathbf{X}_{ii}$ is irreducible or $\mathbf{X}_{ii}=[0]_{1\times 1}$. Finally, if there exist rows having nonzero entries *only* in diagonal blocks, then symmetrically permute all such rows to the bottom to produce

$$\mathbf{T} \sim \left[\begin{array}{cccc|cccc} \mathbf{T}_{11} & \mathbf{T}_{12} & \cdots & \mathbf{T}_{1r} & \mathbf{T}_{1,r+1} & \mathbf{T}_{1,r+2} & \cdots & \mathbf{T}_{1m} \\ \mathbf{0} & \mathbf{T}_{22} & \cdots & \mathbf{T}_{2r} & \mathbf{T}_{2,r+1} & \mathbf{T}_{2,r+2} & \cdots & \mathbf{T}_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{T}_{rr} & \mathbf{T}_{r,r+1} & \mathbf{T}_{r,r+2} & \cdots & \mathbf{T}_{rm} \\ \hline \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{T}_{r+1,r+1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{T}_{r+2,r+2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{T}_{mm} \end{array}\right] \tag{B.1}$$

where each $\mathbf{T}_{11},\ldots,\mathbf{T}_{rr}$ is either irreducible or $[0]_{1\times 1}$ and $\mathbf{T}_{r+1,r+1},\ldots,\mathbf{T}_{mm}$ are irreducible. When the states of a reducible chain $\mathbf{T}$ have been rearranged to be in the form of the right-hand side of (B.1), $\mathbf{T}$ is said to be in *canonical form*. From now on, assume that $\mathbf{T}$ is in canonical form.

A property of irreducible chains that ensures convergence is *primitivity*.

Definition 7. Primitive Matrices

1. A nonnegative irreducible matrix $\mathbf{T}$ having only one eigenvalue $\rho(\mathbf{T})$ on its spectral circle is said to be *primitive*.
2. A nonnegative irreducible matrix $\mathbf{T}$ is primitive if and only if $\lim_{k\to\infty}\mathbf{T}^k$ exists in which case

$$\lim_{k\to\infty}\mathbf{T}^k = \mathbf{G} = \frac{\mathbf{u}\mathbf{v}^\mathsf{T}}{\mathbf{v}^\mathsf{T}\mathbf{u}} > 0$$

where $\mathbf{u}$ and $\mathbf{v}$ are the respective Perron vectors for $\mathbf{T}$ and $\mathbf{T}^\mathsf{T}$ and $\mathbf{G}$ is the *spectral projector* associated with the spectral radius.

The next theorem is a summary of the results for irreducible chains. It complements the results in Proposition 1.

Theorem 8. *Irreducible Markov Chains*

Let $\mathbf{T}_{n\times n}$ be an irreducible stochastic matrix and let π denote the left-hand Perron vector for $\mathbf{T}$.

1. *The kth step transition matrix is $\mathbf{T}^k$.*
2. *If $\mathbf{T}$ is primitive and if $\mathbf{1}$ denotes a column vector of ones, then* $\lim_{k\to\infty}\mathbf{T}^k = \mathbf{1}\pi^\mathsf{T}$.