Promoting a Reputation for Quality*

Daniel N Hauser†

March 23, 2017

Abstract

A firm builds its reputation not only by investing in the quality of its products, but also by controlling the information consumers observe. I consider a model in which a firm invests in both product quality and a costly signaling technology in order to build its reputation, defined as the market's belief that its quality is high. The signaling technology influences the rate at which consumers receive information about quality: the firm can either promote, which increases the arrival rate of signals when quality is high, or censor, which decreases the arrival rate of signals when quality is low. I study how the firm's incentives to build quality and to signal depend on its reputation and current quality. Whether the firm promotes or censors plays a key role in the structure of equilibria. Promotion and investment in quality are complements: the firm has a stronger incentive to build quality when the promotion level is high. Costly promotion can, however, reduce the firm's incentive to build quality as higher quality will lead to higher promotion expenses; this effect persists even as the cost of building quality approaches zero. Censorship and investment in quality are substitutes. The ability to censor can destroy a firm's incentives to invest in quality, because instead of building quality a firm may simply opt to reduce information about low quality products.

KEYWORDS: Reputation, Advertising, Promotion, Censorship, Dynamic Games
JEL: C73, D82, D83, D84

*Thanks to Aislinn Bohren, George Mailath, and Steven Matthews for their advice and support. I'm also grateful for suggestions from Simon Board, David Dillenberger, Mallesh Pai, Alessandro Pavan, Andrew Postlewaite, Philipp Strack, Rakesh Vohra, Yuichi Yamamoto and audiences at the Midwest Economic Theory Conference, North American Summer Meetings, GAMES 2016, the Stony Brook Game Theory Festval, the University of Pennsylvania Theory Lunch, Theory Reading Group, and Empirical Micro Reading Group.

†Email: dahauser@sas.upenn.edu; University of Pennsylvania

1 Introduction

Firms often have a great deal of influence over the information consumers observe about its product. A technology company advertises improvements to its phones, a publisher sends out copies of books to reviewers to make guarantee good reviews, and a tobacco company suppresses research into the negative effects of smoking. In this paper, I allow firms to both invest and strategically manage the information consumers observe about quality. This ability plays a crucial role in the incentive to build and maintain a reputation. A firm's ability to promote positive information about the product is a complement for investment in the quality of the product, and creates persistent reputation effects. In contrast, a firm's ability to censor information is a substitute for investment, and can completely destroy its incentive to build quality. So while a smartphone manufacturer can recover from a bad reputation through a combination of effective promotion and high quality products, a tobacco company that can effectively censor negative information will never be able to maintain a reputation in the long run.

In the model, a long-lived firm sells a product to short-lived consumers. The product's quality changes at a Poisson rate λ, and is fixed between arrivals, as in Board and Meyer-ter-Vehn (2013). Quality is not observed by consumers; instead they observe a Poisson process with an arrival rate that depends on both the quality of the product and the firm's choice of information. Whenever an arrival occurs, consumers observe the quality of the firm's product. The firm's reputation is modeled as consumers' belief about current quality. The firm can promote its product by increasing the intensity of this news process or censor information by decreasing the intensity of this process. Consumer beliefs about past investment and past quality play a crucial role in the firm's current payoff. A consumer is willing to pay a higher price for a product today if she thinks it was a high quality product yesterday, so the firm has incentives to both invest in quality and in controlling the information consumers observe in order to maintain its reputation.

I first consider the good news case, in which a firm selects the arrival rate of news that the product is high quality. In this case, the firm chooses to generate signals only when it is producing a high quality product. This creates incentives for the firm to invest in quality, since the firm knows that it will then be able to promote the product and sell it at a higher price. The ability to promote can create periods of time where reputation cycles, but the firm is solely investing in promotion and not in quality. The ability to control information can also damage incentives and hurt the firm's payoffs. Investing in quality and investing in promotion are complements, in that a firm only benefits from investing in quality if it also promotes. The need to promote makes investing more costly, because even if the firm creates a high quality product, it still has to promote it in order to benefit from the quality breakthrough. The firm is not only negatively impacted by the cost of promotion, but is also damaged by lower consumer beliefs. Endogenous promotion decreases a firm's incentives to invest in quality, which in turn leads to lower consumer beliefs and lower payoffs for the firm. Investment in quality is under-incentivized. Even as investing in quality becomes arbitrarily cheap the firm still shirks when it has a high reputation.

I focus on Markov Perfect Equilibrium (MPE) taking the firm's reputation as the state variable. By focusing on MPEs, I can investigate how the firm's reputation creates incentives for investment in quality and information. Every MPE of the good news game is characterized by three regions. When consumer beliefs are high, the firm chooses not invest in quality or promotion. At intermediate beliefs, the firm promotes if it is selling a high quality product but chooses not to invest in quality. In this intermediate region, the firm invests in promotion to restore its high reputation. Finally, at low beliefs, the firm promotes when it has a high quality product, and the firm invests in improving or maintaining quality, independent of the quality of the product it is currently selling. The firm will never invest in quality unless it also has incentives to promote, because there is no way for consumers to detect or punish low levels of investment if they do not expect to see. This equilibrium structure is distinct from the equilibrium in Board and Meyer-ter-Vehn (2013) and Marinovic, Skrzypacz, and Varas (2015), who consider a similar model. In particular, the intermediate region where the firm invests in promotion but not in quality does not exist in either Board and Meyer-ter-Vehn (2013) (who take promotion as exogenous) or Marinovic, Skrzypacz, and Varas (2015). This region allows the firm to profit from promotion, which the firm never does in any MPE in Marinovic et al. (2015), and leads to different long-run belief dynamics than the dynamics in Board and Meyer-ter-Vehn (2013).

This insight applies to many economic settings. A smartphone manufacturer initially invests heavily in both the quality of its new phone and in promoting it when its reputation is very low. Once established, the firm only makes minor improvements to newer versions of the phone, yet it still spends a lot on advertising newer versions. In the good news case, the firm utilizes its ability to control promotion and the quality of its product to create and maintain its reputation. Even though the firm is only investing in maintaining quality at very low reputations, the firm's reputation is still persistent, the firm will constantly invest in renewing its reputation through promotion and occasional investment in quality.

Next, I consider the bad news case, where bad news about the product arrives at a fixed rate, and the firm can suppress this information. Censorship and investment are substitutes. A firm can reduce bad news by either investing in quality or censoring the news. This can cause incentives to disintegrate. The firm is often willing to forgo investment in quality, because it knows censorship is a viable alternative. Oil companies, the food industry, and cigarette manufacturers are prime examples. Instead of investing in better, safer products, they have invested significantly in suppressing and undermining research that proves their products are harmful. For instance, the Sugar Research Foundation paid three scientists $50,000 to downplay research that showed a connection between heart disease and sugar.[1] There is significant evidence tobacco industry knew as early as 1963 about the hazards of cigarettes, but hid this research from the public, and suppressed further research.[2][3] Fossil fuel producers like Exxon knew about the harmful effects of fossil fuels over 10 years before it became public knowledge, and they actively

[1]See http://www.nytimes.com/2016/09/13/well/eat/how-the-sugar-industry-shifted-blame-to-fat.html

[2]See http://www.nytimes.com/1994/05/07/us/tobacco-company-was-silent-on-hazards.html

[3]See http://www.nytimes.com/1994/04/29/us/scientists-say-cigarette-company-suppressed-findings-on-nicotine.html

worked to hide this from the public.[4]

In the bad news case, any MPE can be characterized by two reputation cutoffs. The firm invests in quality when its reputation is higher than a cutoff and doesn't invest otherwise. Similarly, the firm censors bad news when its reputation is sufficiently high, and doesn't otherwise. When censorship is cheaper than investment, the firm opts not to invest in quality. Instead, it opts to censor bad news when its reputation is high. Unless investment in quality is sufficiently less expensive than censorship, the firm will always substitute censorship for investment in quality, and therefore will never invest in the quality of its product. In these equilibria, the firm's reputation is transient. Consumers have no reason to believe the firm is investing in its product, and therefore are skeptical of the product. This creates a situation in which the firm would prefer to commit to never censor, and, in fact, may benefit when censoring becomes more costly.

Finally, I consider the case of exogenous news. In the good news case, there is some exogenous rate that good news arrives at, absent promotion, and in the bad news case, the firm's ability to censor is imperfect. Bad news can never be shut down completely. In the good news case, if the cost of investment is high enough relative to the cost of promotion and the speed of exogeneous news, then the equilibria have the same structure as in the good news case without exogenous news. But if the cost of investment is relatively low, then in the long run, the firm's reputation dynamics converge to the long run dynamics of the equilibrium in Board and Meyer-ter-Vehn (2013). In the bad news case, the result that the firm never invests in quality is robust to the firm being unable to shut down bad news completely, and to the addition of small amounts of endogenous or exogenous good news.

Literature Review

This paper builds on the reputation model of Board and Meyer-ter-Vehn (2013), in which a firm invests in its product's quality to build a reputation. Quality shocks arrive at a Poisson rate, and the quality of a product after a shock depends on the firm's current effort. They characterize how a firm builds and maintains a reputation for selling a high quality product. The firm's reputation dynamics depend crucially on the information structure. In the good news case, the firm's reputation cycles. After good news arrives, the firm allows both its reputation and its quality to decay by choosing not to invest and then resumes investment when reputation is low to build it back up. On the other hand, in the bad news case, the firm's reputation is path dependent; the firm's incentives increase as the firm's reputation becomes higher, so if the firm's reputation ever becomes low enough, the firm can never recover. Board and Meyer-ter-Vehn (2015) extend this model to study a lifecycle model where the firm can choose to when to exit the market. Halac and Prat (2014) use this framework to study the problem of a manager that is trying to build a reputation for being attentive.

[4]See https://www.scientificamerican.com/article/exxon-knew-about-climate-change-almost-40-years-ago/

In related work Marinovic, Skrzypacz, and Varas (2015). They consider a similar environment, where quality of the product evolves as in Board and Meyer-ter-Vehn (2013), but the firm can now deterministically reveal its quality at any instant of time, for a cost. They find that every Markov Perfect Equilibrium has many counterintuitive features. Making it cheaper for the firm to reveal its type always lowers payoffs, and the firm almost never benefits from a higher reputation. They resolve this issue by showing that by using time as a state variable, they can significantly enrich the state space, and support higher payoffs through a more elaborate system of punishments and rewards. My result illustrates that the counter-intuitiveness of the MPE in their paper is a result of their modeling assumptions, and do not hold when the firm can't deterministically reveal its type. In contrast, my paper establishes that the firm can (and will) invest in promotion and not quality at some reputations, and its ability to promote allows the firm to extend the period it collects reputational dividends. This cannot happen in Markov Perfect Equilibria in Marinovic et al. (2015).

In addition to Board and Meyer-ter-Vehn (2013), several other papers model reputation as beliefs over a persistent state, such as product quality, that evolves endogenously over time. Cisternas (2015) considers a career concerns model where the worker's type evolves over time, and the worker can exert effort to try to mimic higher or lower skilled workers. The worker's quality is persistent, which gives the firm persistent incentives to exert effort. Bohren (2012) considers an application in which a firm's current quality is partially determined by a persistent, observable state variable that noisily depends on the firm's past effort choices. One can view the state as the firm's reputation, as it informs consumers' beliefs about the firm's current quality. This persistence gives the firm an incentive to invest in quality. Whether the firm continues to invest in quality at high states depends on the boundedness of the return to quality.

There is an extensive different, but related, literature that models reputation as an adverse selection problem, starting with Kreps, Milgrom, Roberts, and Wilson (1982), and continuing in papers like Fudenberg and Levine (1989) and Cripps, Mailath, and Samuelson (2004). These papers model reputation as the consumer's belief about the firm's type, which can either be a strategic type or a behavioral type. In these papers, players do not care about the firm's type directly, instead they care about the strategic behavior that is induced by the possibility of the firm being a behavioral type. Firms can credibly take actions they wouldn't take in the absence of reputation effects in order to influence the inference other players make about their type. These reputational concerns allow the firm to achieve payoffs that exceed the payoffs of the game where the consumers know the firm's type.

There is also a literature on designing an information structure to encourage reputation building. Dellarocas (2005) considers an environment where a long-lived seller selling to short-lived buyers has a moral hazard problem, and the designer designs a signal to help resolve this problem. This persistent signal improves the seller's incentives to exert effort, since she will be rewarded in future periods for high values of the signal, and punished for low values. Hörner and Lambert (2015) consider a similar problem in a career concerns environment and in addition allow the designer to choose multiple signals. In these papers, a precise choice of information structure can reduce the moral hazard problem. Finally, Pei (2015) considers how a firm can control the information the market sees about a worker in order to keep the worker's outside option low. He finds that the firm's ability to hide information about their workers reduces the firm's ability to induce high levels of effort.

One can view promotion in my paper as a form of advertisement. This is similar in some ways to the informational story of advertising, explored in a large literature that dates back to Nelson (1974) and Milgrom and Roberts (1986). These papers model advertising as a signaling game, where consumers make inference based on the firm's expenditure on advertising. Firms with high quality products have lower costs of advertising, and thus are able to launch more extravagant promotional campaigns. I explore a different aspect of informational advertising. Consumers do not observe advertising expenditure directly, or make inferences based on it. Instead, advertising speeds up how fast consumers learn the quality of the product, which captures features not only of traditional advertising, but also promotional campaigns like sending preview copies of books to reviewers, supporting or suppressing research, and encouraging good reviews. The model considered in this paper does not allow for the possibility that firm can manipulate the informational content of news. For instance, in the signaling story a firm with a low quality product can pretend to be a firm with a high quality product, which cannot happen in this model.

The rest of this paper proceeds as follows. Section 2 outlines the model. Section 3 presents the analysis the good news case, including the characterization of equilibria, and comparisons to the the equilibria in Board and Meyer-ter-Vehn (2013) and to the equilibria in the game in which the firm can commit to an information structure. Section 4 presents the analyze the bad news case, and demonstrates how the ability to censor can lead to a breakdown of the firm's incentives to invest in quality. Finally, section 5 considers the addition of exogenous news, and what impact this has on the structure of equilibria. All proofs are in the appendix.

2 Model

The Firm

Time is continuous, $t \in [0, \infty)$. There is a single long lived firm with stochastic quality $\theta_t \in \{L, H\}$. The firm has discount rate r. At each instant of time, the firm chooses an action, which consists of a level of effort $a_t \in [0, 1]$ and a level of promotion $\pi_t \in [0, \bar{\pi}]$, for costs ca_t and $k\pi_t$, where $c > 0$, $k > 0$ and $\bar{\pi} > 0$. The upper bound on promotion, $\bar{\pi}$, is fixed and measures the maximum arrival rate of news generated by the firm. A firm's strategy is a stochastic process $(a_t, \pi_t)_{t=0}^{\infty}$ that determines the effort choice and level of promotion at each instant of time given the history the firm has observed. These strategies are predictable processes with respect to the σ-algebra generated by the quality Poisson process and the news Poisson process. Quality evolves via Poisson shocks, as in Board and Meyer-ter-Vehn (2013). Specifically, there is a Poisson process with intensity

$\lambda > 0$. Whenever there is an arrival of this process, θ_t becomes H with probability a_t and L with probability $1 - a_t$, and is fixed between arrivals. The firm observes θ_t.

Consumers

Consumers do not observe θ_t, a_t or π_t directly. Consumers observe a Poisson process, which provides a noisy signal that they use to form beliefs about θ_t. I consider two processes, good news and bad news. This news process has intensity $\pi_t 1_{\theta_t = H}$ in the good news case, and intensity $(\bar{\pi} - \pi_t)1_{\theta_t = L}$ in the bad news case. Let $x_t = Pr(\theta_t = H | \mathscr{F}_t^s)$, where $\mathscr{F}_t^s$ is the σ-algebra generated by the news process, and the probability measure is the measure induced by the consumers' beliefs about the firm's strategy.

In the good news case, news can only arrive when the firm is selling a high quality product, and in the bad news case news can only arrive when the firm is selling a low quality product. When a consumer sees news, she makes a positive inference in the good news case and a negative inference in the bad news case. The firm's choice of π speeds up the arrival rate of good news and slows down the arrival rate of bad news.

Payoffs

The firm receives a flow payoff of x_t. This can be motivated either as the willingness to pay of consumers with utility $1_{\theta_t = H}$ or as consumers who are willing to pay 1 arriving at some rate proportional to the public belief. The firm maximizes

$$\max_{\hat{a}, \hat{\pi}} E_{\hat{a}, \hat{\pi}} \left(\int_0^\infty e^{-rt} [x_t - c\hat{a}_t - k\hat{\pi}_t] dt \right)$$

where the expectation is taken with respect to the actual probability measure induced by the firm's chosen effort and promotion levels, while x_t is determined by what consumers believe about the firm's effort and promotion choices.

Solution Concept

I characterize Markov Perfect Equilibrium. These are equilibria where strategies only condition on consumers' beliefs and current quality.

Definition 1 *A pure strategy Markov Perfect Equilibrium (MPE) consists of a markov effort strategy $a : [0,1] \times \{H, L\} \to [0,1]$, which maps public beliefs to effort, and a markov promotion strategy $\pi : [0,1] \times \{H, L\} \to [0, \bar{\pi}]$ and believed markov strategies $\tilde{a}$ and $\tilde{\pi}$ such that:*

1. *(a, π) are sequentially rational.*

2. *Beliefs x_t are formed through Bayes Rule given believed strategies $(\tilde{a}, \tilde{\pi})$.*[5]

3. *Beliefs are correct, $a = \tilde{a}$, $\pi = \tilde{\pi}$.*

An additional admissibility restriction must be placed on beliefs in order to ensure the belief process has a unique solution in the MPE. These restrictions are identical to the restrictions placed in Board and Meyer-ter-Vehn (2013). If beliefs follow the law of motion $\dot{x}_t = g(x_t)$, then the drift $g(x)$ must satisfy one of the following conditions:

1. $g(x) = 0$
2. $g(x) > 0$ and $g(x)$ is right continuous at x,
3. $g(x) < 0$ and $g(x)$ is left continuous at x,

at any cutoff and beliefs can be partitioned into a finite set of intervals such that both the effort and promotion choices are Lipschitz continuous on the interior of all these intervals.

These restrictions are placed on consumer's beliefs about the firm's strategies, not on the firm's strategies themselves. Admissibility ensures that $\dot{x} = g(x)$ has a solution and when there are multiple solutions, I select the one consistent with the discrete time approximation (for details, see Board and Meyer-ter-Vehn (2013) or Klein and Rady (2011)). As in Board and Meyer-ter-Vehn (2013), this is a relatively mild assumption that ensures beliefs are defined everywhere and are right continuous when viewed as a function of t.

Discussion of the Model

There are many ways to interpret the quality process. For instance, arrivals can be viewed as the firm's ability to incorporate new discoveries, a firm investing in keeping a valued employee, or as the firm purchasing a rate of technological improvement or abating deterioration of the firm's technology. While the technology process is stylized, it satisfies many intuitive properties of persistent quality. Beliefs drift down when the firm is believed to be shirking and up when the firm is believed to be working absent any promotion, and the firm's incentives to exert effort depend both on present and future quality.

The news process is similarly stylized. The assumption that arrivals of news can only occur when the firm has a high quality product in the good news case, or when the firm has a low quality product in the bad news case is a strong assumption. This can be viewed

[5] While there may be off path actions in some cases considered in this paper, off path beliefs are completely pinned down by Bayes Rule. If news that is believed to occur with 0 probability observes, beliefs are restricted to be consistent with the quality revealed by that news and beliefs subsequently updates through Bayes Rule.

as a requirement that a successful promotional campaign contain evidence that the firm's product is actually good, and that a negative news story needs evidence that the firm's product is actually bad. The Poisson structure captures large jumps in beliefs about quality. For example, an actor winning an academy award after a production company's promotional campaign or a news story about a popular brand of tires exploding causes beliefs to jump significantly. This process abstracts away from infinitesimal information consumers receive constantly about a product. For instance, a consumer continuously updating beliefs about while they are using a product. This information seems more difficult for the firm to control, although a model with brownian exogenous news may be an interesting extension.

This information process and $\bar{\pi}$, the arrival rate of news, are important objects in this model and the analysis. $\bar{\pi}$ can be viewed as a measure of how difficult it is to successfully create an ad campaign (or some other sort of signal) that convinces consumers that the firm indeed is selling a high quality product or a measure of how quickly consumers see an ad campaign. This upper bound encapsulates the difficulty of creating a promotional campaign; how long it takes for the campaign to permeate the public consciousness; and the difficulty of actually persuading consumers. Therefore, it is in some ways a measure of persuasiveness.

3 The Good News Case

In the good news case, the signal process has intensity $\pi_t 1_{\theta_t=H}$. Arrivals can only occur when the firm has a high quality product, in which case, beliefs jump to 1. Between arrivals, beliefs follow the law of motion

$$\dot{x}_t = \underbrace{\lambda(\tilde{A}_t - x_t)}_{\text{Quality Breakthroughs}} - \underbrace{\tilde{\Pi}_t x_t(1 - x_t)}_{\text{Absence of signals}},$$

where $\tilde{A}_t = E(\tilde{a}_t|\mathcal{F}_t)$ is the believed amount the firm is investing in quality and $\tilde{\Pi}_t = \pi_t 1_{\theta_t=H}$ is the believed level of promotion.

In between arrivals, the drift of consumer's beliefs can be expressed as the sum of two terms. The firm's effort choice determines the first term $(\tilde{A}_t - x_t)\lambda$. Depending on how much effort the firm is believed to be exerting, this term is either positive or negative. If consumer's believe it is more likely that high quality products are switching to low quality products then low quality products are switching to high quality products, then it is negative, otherwise it is positive. The second term, $-\tilde{\pi}_t x_t(1 - x_t)$ is determined by the firm's choice of promotion. This term is negative if consumers believe the firm is promoting, since it is more likely that no news arrives if the firm is selling a low quality product, and 0 if the firm is not promoting.

The firm's value function is

$$V(x, \theta) = \max_{a,\pi} E\left(\int_0^\infty e^{-rt}[x_t - ca_t - k\pi_t]dt \middle| \theta_0 = \theta, x_0 = x\right).$$

This can be rewritten as

$$V(x_0,\theta)=\max_{a,\pi}\int_0^\infty e^{-\int_0^t(r+\pi(x_s,\theta)+\lambda)ds}[x_t+a(x_t,\theta)(\lambda(V(x_t,H)-V(x_t,L))-c)+\lambda V(x_t,L)$$
$$+\pi(x_t,\theta)1_{\theta=H}(V(1,\theta)-k)]dt,$$

where x_t is the solution to $\dot{x}_t=\lambda(\tilde{a}_t-x_t)-\tilde{\pi}_t x_t(1-x_t)$. As in Board and Meyer-ter-Vehn (2013), this can be rewritten as

$$V(x_t,\theta)=\max_{a,\pi}\int_0^\infty x_t+a(x_t,\theta)(\lambda D(x_t)-c)+\pi(x_t,\theta)(\Delta(x_t)-k)+\lambda V(x_t,L)-\lambda V(x_t,\theta)-rV(x_t,\theta)dt,$$

where $D(x)=V(x,H)-V(x,L)$ and $\Delta(x)=V(1,H)-V(x,H)$. This implies the following lemma.

Lemma 1 *(**Sequential Rationality**) (a,π) are a MPE if and only if $a(x,\omega)$ solves*

$$\max_{a\in[0,1]}\lambda D(x)a-ca,$$

$\pi(x,H)$ solves

$$\max_{\pi\in[0,\bar{\pi}]}\Delta(x)\pi-k\pi,$$

and $\pi(x_t,L)=0$ for all x.

The firm's incentive to exert effort is driven by $D(x)$, the change in the firm's expected payoffs if a quality change arrives at that instant. The firm's incentive to promote is driven by $\Delta(x)$; the change in payoffs if a beliefs jumped at that instant. An important feature of this model is that the effort choice is independent of the firm's type, which greatly simplifies the analysis.

$D(x)$ and $\Delta(x)$ have a tight relationship, because the increase in a firms payoff from having high quality is due to the potential reputational benefits the firm receives in the future once it is selling a high quality product. This is captured in the following proposition.

Lemma 2 *The difference in expected payoffs between a firm selling a high quality product and a firm selling a low quality product can be written as*

$$D(x_0)=\int_0^\infty e^{-(r+\lambda)t}\pi(x_t,H)[\Delta(x_t)-k]dt.$$

Moreover, payoffs satisfy the following properties for any beliefs

1. *$V(\cdot,H)$ and $V(\cdot,L)$ are strictly increasing.*
2. *$V(x_t,H)\geq V(x_t,L)$ for all x_t.*
3. *$\Delta(x)$ is decreasing.*

4. $D(x)$ is decreasing.

The value functions are increasing. Given any two initial conditions x_0 and x_0^*, if $x_0 > x_0^*$ then $x_t > x_t^*$ until the first arrival. The firm can then mimic the $*$-firm and induce the same probability measure over θ_t and arrivals at all points, while receiving a strictly higher flow payoff. Therefore, the its payoff must be higher. Similarly, $V(x_t, H) \geq V(x_t, L)$ because a high quality firm can play the exact same strategy as the low quality firm to guarantee themselves the same payoff. This directly implies that $\Delta(x)$ is decreasing, and this, combined with the optimality of π and the previous observation about x_t and x_t^*, implies that $D(x)$ is decreasing.

The previous lemmas imply that optimal strategies are cutoff strategies. The firm shirks and doesn't promote at high beliefs and as beliefs drift down eventually starts promoting and working. This greatly simplifies the existence argument and implies that in any equilibrium beliefs can be categorized into four different regions; the region where the firm is promoting or working, the region where the firm is promoting but not working, the region where the firm is working but not promoting and the region where the firm is working and promoting. The following proposition establishes the unique structure of any Markov Perfect Equilibrium.

Proposition 1 *Every Markov Equilibrium is characterized by two cutoffs $x^*, x^{**} \in [0, 1)$, such that $x^* \leq x^{**}$. Optimal strategies satisfy*

$$a(x) = \begin{cases} 1 \text{ if } x < x^* \\ 0 \text{ if } x > x^* \end{cases},$$

$$\pi(x, H) = \begin{cases} \bar{\pi} \text{ if } x < x^{**} \\ 0 \text{ if } x > x^{**} \end{cases},$$

and $\pi(x, L) = 0$. When $\bar{\pi} \leq \lambda$

- *$x^* < x^{**}$ in any MPE where $x^{**} > 0$.*
- *a MPE exists.*

As beliefs decrease, the firm always begins promoting before it starts working. If there were a region where consumers believed the firm was working and not promoting, their beliefs would drift up, so it would never have any incentive to promote once beliefs were high enough. But if this was the case, then the firm would have no incentive to work, since the information consumers expect to see wouldn't change if the firm shirked. Therefore the firm must start promoting before it starts working. This can lead to long cycles where the firm builds and maintains reputation through promotion without exerting effort. After a quality breakthrough and a successful promotion, the firm collects reputational dividends and tries to use its ability to promote to extend the period of time when it collects these dividends without investing in quality.

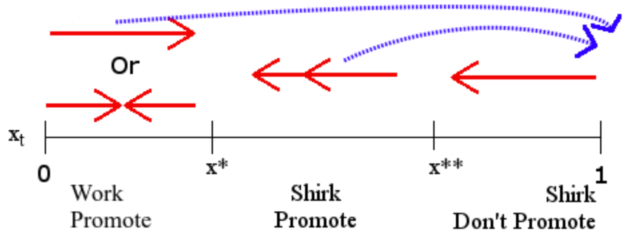

Figure 1: Structure of a MPE

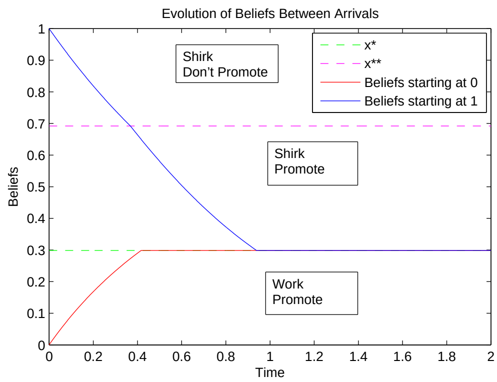

Figure 2: Belief Dynamics between arrivals

Markov Perfect Equilibria always exist if $\lambda \geq \bar{\pi}$. From a technical perspective, this assumption implies that beliefs are drifting up when the firm is investing and down when the firm isn't, which implies that at x^* the drift is always 0 at any equilibrium. This assumption means that the quality of the firm's product changes faster than the firm can signal that it has a high quality product. This is a natural assumption for a variety of applications. For instance, an athlete's ability is relatively responsive to how much effort they exerted at practice that season, while our beliefs about his or her quality change relatively infrequently. Tech firms are constantly innovating, changing both hardware and software for a new phone, while our beliefs that the next phone is going to be good or bad seem to change relatively slowly. When a firm stops investing in its star employees, they leave and find new employment much faster than the firm can credibly produce a evidence of high quality, in particular in fields where the market for star employees is particularly competitive.

The result that the firm never exerts effort before it starts promoting is also present in Marinovic et al. (2015). But, unlike in Marinovic et al. (2015), this is not always bad for the firm, since beliefs no longer jump to 0 as soon as consumers believe the firm would be promoting but don't see any news. In Marinovic et al. (2015), promotion never has any value to the firm in equilibrium. The best possible case is that the marginal benefit from promotion is equal to its cost. As opposed to the MPEs in Marinovic et al. (2015), in this paper the firm exerts effort when beliefs are non-zero and benefits from consumers giving them the benefit of the doubt when they don't see any news.

I consider two different types of commitment to investigate how endogenous promotion changes firm payoffs. I first examine how the game with endogenous promotion compares to the game with exogenous promotion from Board and Meyer-ter-Vehn (2013). In this game, the firm pays a lump sum at the beginning of the game in order to commit to promoting whenever it has a high quality product for no flow cost.[6] I then consider what strategy promotion strategy the firm would like commit in the game with endogenous promotion. Since the firm's promotion strategy was not observable by consumers, the firm is unable to fully internalize the impact its choice of promotion has on the drift of beliefs and on the level of investment. By allowing the firm to commit to a promotion strategy, I can investigate whether the firm is over or under-investing in promotion.

The ability to promote creates a moral hazard problem. The firm needs to be promoting before it has any incentive to invest in its product. Consider the alternative game from Board and Meyer-ter-Vehn (2013), where the firm cannot control the news process. Instead consumers learn by observing an exogenous news process with arrival rate $\bar{\pi}1_{\theta_t=H}$.

Board and Meyer-ter-Vehn (2013) show that in the game with exogenous promotion, all MPEs have the following form

$$a(x,\theta) = \begin{cases} 1 \text{ if } x < x^*_{BM} \\ 0 \text{ if } x > x^*_{BM} \end{cases}$$

for some cutoff $x^*_{BM} \in [0,1]$. Comparing this cutoff to the cutoff in when the firm can endogenously promote allows me to look at how the ability to endogenously choose the level of promotion changes how reputational concerns incentivize the firm to invest in quality.

Proposition 2 *If $\lambda \geq \bar{\pi}$ then the equilibrium of the game with exogenous promotion is unique and $x^*_{BM} \geq x^*$ (and is strict as long as $x^*_{BM} \neq 0$). Let $\mathcal{X}$ be the stationary distribution, supp$(\mathcal{X}_{BM}) \subseteq$ supp$(\mathcal{X})$.*

The firm's incentives to invest in quality are reduced when compared to the game studied in Board and Meyer-ter-Vehn (2013). While the firm still eventually invests in quality, it starts investing at lower reputations, so the incentive to exert effort to avoid a bad reputation is diminished. The ability to choose when to promote decreases the gap between the payoffs received by a firm selling a high quality product and a firm selling a low quality product, so the firm chooses to start working at a lower belief. This does not necessarily imply the firm is always investing less in quality, because beliefs move in a fundamentally different way in the two models. While the firm with exogenous promotion starts working at a higher belief, it expect beliefs to jump to 1 more, since arrivals can occur anywhere, and these arrivals delay when it has to begin investing in effort. On the other hand, when the firm has a low quality product, it will begin investing sooner in the game with exogenous promotion than in the game with endogenous promotion, because no news can arrive to delay investment.

The firm receives some benefit from exogenous promotion. For instance, a firm selling a low quality product when beliefs are close to 1 receives higher flow payoffs for longer

[6] If promotion still has a flow cost, the firm may never want to invest in quality because it doesn't want to pay for promotion.

because consumers' no longer make inferences from not seeing any news, but there also is a cost. After news arrives, consumer beliefs can drop to a lower point than it would in the game with exogenous promotion, which can lower the firm's payoffs. This suggests that a firm may benefit from committing to exogenous promotion by hiring an outside advertising agency, in order to incentivize effort at higher beliefs.

When promotion is exogenous, consumers learn that that the firm is selling a high quality product more frequently.

Corollary 1 *Let $x_0 = 1$, and $\lambda \geq \bar{\pi}$. Let $\tau = \inf\{t > 0 : x_t = 1\}$, and $\tau_{BM} = \inf\{t > 0 : x_{BM,t} = 1\}$. Then $E(\tau) > E(\tau_{BM})$.*

The expected time it takes for reputation to return to 1 is smaller when the firm has committed to exogenous promotion. Reputation cycles more frequently when the firm has committed to promote. The increased incentive to invest leads to a higher shirk-work cutoff x^*, which in turn leads to consumers learning that the firm is selling a high quality product more frequently.

This suggests that the firm may benefit overall from committing to promote whenever it has a high quality product, independent of its reputation. If the firm could pay an advertising agency a lump sum in exchange for the ad agency committing to promote, they would benefit from the enhanced incentives for investment in quality. This is explored more in the following section, in the limiting case as $c \to 0$.

In equilibrium, the firm is not fully internalizing the impact of its choice on promotion on the law of motion for beliefs and on its choice of investment. To explore this effect more, suppose the firm could commit to any cutoff x^{**}_{FB} ex-ante. With this commitment power, the firm internalizes its ability to control the drift of beliefs, and how its choice determines what investment strategies are credible.

Proposition 3 *Suppose $\lambda \geq \bar{\pi}$. If the firm could commit to promoting below any cutoff, it would commit to promoting at a cutoff $x^{**}_c \leq x^{**}$ for any x^{**} that is a promotion cutoff in a MPE of the original game.*

Given an equilibrium, with cutoffs x^*, x^{**}, in the game where the firm has committed to the optimal promotion strategy, it chooses to invest in quality at a higher reputation, $x^*_c \geq x^*$ for any x^* that is a investment cutoff in the MPE of the original game if $\theta_0 = H$ and $x_0 \in [0, 1] \setminus [x^*, x^{**})$.

If $\theta_0 = L$, there exists some $\bar{x} > 0$ such that if $x_0 \leq \bar{x}$ the previous result holds.

The firm commits to a higher level of promotion because it now internalizes the impact its promotion has on the drift of beliefs. This in turn, can increase the incentives to invest, by raising the payoff from a high reputation. The firm is under-investing in quality and over-investing in promotion. The firm would like to be promoting less, in order to prevent consumers from interpreting lack of news as a bad sign, but without commitment power, it cannot convince consumers that it wouldn't promote at a higher reputation.

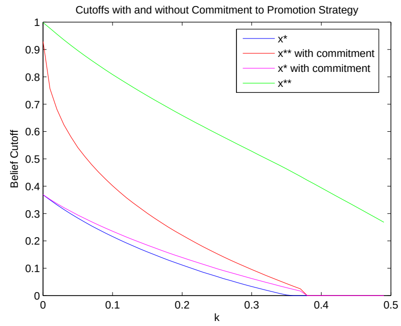

Figure 3: Optimal level of investment and promotion with and without commitment.

In order to capture how the firm's ability to promote impacts incentives, it is interesting to consider the limiting case as $c \to 0$. This simplifies the firm's effort choice, and allows me to focus on the firm's promotion decision and how that decision impacts payoffs. In addition, while there may be multiple equilibria in this model, these equilibria all converge to the same limiting equilibrium as $c \to 0$, which allows me to characterize some comparative statics.

Proposition 4 *As $c \to 0$, if $\lambda \geq \bar{\pi}$,*

1. $|x^* - x^{**}| \to 0$
2. *There is a unique point $\bar{x}^{**}$ such that for any sequence of costs $(c_n)_{n=1}^{\infty}$ and a corresponding sequence of equilibria, characterized by cutoffs $(x_n^*, x_n^{**})_{n=1}^{\infty}$, $x_n^{**} \to \bar{x}^{**}$. moreover $\bar{x}^{**} < 1$.*
3. *In the limiting equilibrium, as k decreases payoffs increase.*

Unlike in Marinovic et al. (2015), decreasing k doesn't always decrease firm payoffs. When costs are sufficiently low, the firm benefits from cheaper promotion, because it increases the firm's incentives to work. The gap between the work cutoff and the promotion cutoff vanishes, but the firm still only works after it has begun promoting, even when costs are arbitrarily low, since the firm can never credibly work unless consumers expect it to also promote when it has a high quality product. Even as investment becomes arbitrarily cheap, the firm still doesn't work everywhere. The firm still has to pay for promotion to make effort worthwhile, which indirectly makes effort costly.

Reputation cycles persist, even as investment becomes arbitrarily inexpensive. The firm's reputation drifts down until hitting x^*, and the firm then invests in building it back up. Even a very established firm has to invest in promoting its product.

In this limit, I can compare the payoffs from the game with exogenous promotion to the game with endogenous promotion.

Proposition 5 *As $c \to 0$, if $\lambda \geq \bar{\pi}$, $V_{BM}(x,\theta) > V(x,\theta) + E\left(\int_0^\infty e^{-rt} k\pi_t | x_0 = x, \theta_0 = \theta\right)$, where V_{BM} is the value function from the game where news arrives at exogenous rate $\bar{\pi} 1_{\theta_t = H}$.*

So the firm would be willing to pay more than its total expected expenditure on promotion in order to commit to promoting at rate $\bar{\pi}$ whenever it has a high quality product. In this limit, the firm is hurt by the ability to control promotion because it leads to worse consumer beliefs. The firm benefits from hiring an advertising agency, even if the agency doesn't have any sort of comparative advantage or superior technology. The firm benefits simply because the ad agency can be used as a commitment device, convincing consumers that the firm will promote whenever it has a high quality product.

The ability to promote also can lower the firm's payoffs by causing beliefs to drift down faster. If the firm could commit to never promote, beliefs would always follow the law of motion $\dot{x}_t = \lambda(\tilde{a}_t - x_t)$ instead of drifting down at rate $\dot{x}_t = \lambda(\tilde{a}_t - x_t) - \tilde{\pi}_t(x_t)(1 - x_t)$, but the firm also loses the ability to benefit from investing in quality, since consumers no longer see any news. In the limit, as $c \to 0$, I can characterize which of these effects is larger.

Corollary 2 *As $c \to 0$, if $\lambda \geq \bar{\pi}$, payoffs in the limiting equilibrium are greater than payoffs in the equilibrium where the firm has committed to never promote.*

The firm still does benefit from promotion, even though it makes effort costly. Since promotion allows consumers to distinguish between high and low quality firms, it gives firms the ability to build a reputation. Without this promotion technology, the firm would have no incentive to work because shirking would be undetectable. These results contrast with Marinovic et al. (2015) where in any Markov Perfect Equilibria, payoffs are decreasing as k decreases and the firm would rather commit to not promoting in any MPE. The deterministic nature of the firm's signaling technology in Marinovic et al. (2015) model forces the firm to signal so frequently that the firm no longer benefits at all from building a reputation.

4 The Bad News Case

In this section I consider the case in which consumers learn= about the firm's quality through arrivals of bad news, and the firm tries to censor this bad news. Unlike promotion, censorship is a substitute for effort and can cause incentives to completely break down.

The difference between the good news and bad news model is the news process. Now the news process has arrival rate $\max(0, \bar{\pi}1_{\theta_t = L} - \pi_t)$ and $\pi_t \in [0, \bar{\pi}]$. Consumer beliefs now follow the law of motion

$$\dot{x}_t = \underbrace{\lambda(\tilde{A}_t - x_t)}_{\text{Quality breakthroughs}} + \underbrace{(\bar{\pi} - \tilde{\Pi}_t)x_t(1 - x_t)}_{\text{Absence of news}}.$$

Unlike in the good news case, the signaling term now causes beliefs to drift up. If consumers believe the firm is not censoring, not seeing any news is more likely to occur when the firm is selling a high quality product. When news arrives, consumers learn that $\theta_t = L$ and adjust their beliefs to 0.

Incentives to work and censor are now driven by the firm's fear of consumers figuring out that it is selling a low quality product. As before, the firm's value function can be written as

$$
\begin{aligned}
V(x,\theta) = \int_0^\infty e^{-(\int_0^t r+\lambda+\bar{\pi}-\pi_s ds)}[x_s + \bar{\pi}1_{\theta_t=L}(V(0,L) - V(x_t,L)) \\
+ \pi_t(V(x_t,L) - V(0,L) - k)) \\
+ a_t(\lambda(V(x_t,H) - V(x_t,L)) - c) \\
+ \lambda(V(x_t,L) - V(x_t,\theta)) - rV(x_t,\theta)]dt.
\end{aligned}
$$

The firm's incentives to invest in effort and promotion are similar to before. The incentive to invest is still driven by $D(x)$, but now the incentive to promote is driven by $\Delta(x) = V(x_t,L) - V(0,L)$.

Lemma 3 ***(Sequential Rationality)*** *The optimal $a(x_t,\theta)$ solves*

$$\max_{a\in[0,1]} \lambda D(x_t)a - ca$$

and the optimal $\pi(x_t,L)$ solves

$$\max_{\pi\in[0,\bar{\pi}]} \Delta(x_t)\pi - k\pi.$$

This leads to very different effort choices than the good news case. Now the firm has incentives to invest when beliefs are high, because it is worried about consumers finding out that it has a low quality product. Like before, $D(x)$ and $\Delta(x)$ are related.

Lemma 4 *The difference in expected payoffs between a firm selling a high quality product and a firm selling a low quality product can be written as*

$$D(x_0) = \int_0^\infty e^{-(r+\lambda)t}[(\bar{\pi} - \pi(x_t,L))(\Delta(x_t)) - k\pi(x_t,L)]dt.$$

Moreover, payoffs satisfy the following properties for any beliefs

1. *$V(\cdot,H)$ and $V(\cdot,L)$ are strictly increasing.*
2. *$V(x_t,H) \geq V(x_t,L)$ for all x_t.*
3. *$\Delta(x)$ is increasing.*
4. *$D(x)$ is increasing.*

$D(x)$ is increasing in x by similar logic as the good news case. $\Delta(x_t)$ is clearly increasing in x, and π_t maximizes pointwise $\Delta(x_t)\pi_t - k\pi_t$. This is equivalent to minimizing $(\bar{\pi} - \pi_t)\Delta(x_t) + k\pi_t$. Since $\Delta(x_t)$ is strictly increasing in x, so the minimum value of this is increasing in x. Moreover, as before, beliefs that start out higher always stay higher, so for any $x > x'$, the integrand is always greater for x_t than for x'_t, so $D(x) > D(x')$. This means, as before, equilibrium can be characterized in terms of cutoffs x^* and x^{**}.

Proposition 6 *A MPE exists. Moreover, in any MPE, there exist cutoffs $x^* \in [0,1]$ and $x^{**} \in (0,1]$ such that equilibrium strategies take the form*

$$a(x,\theta) = \begin{cases} 1 \text{ if } x > x^* \\ 0 \text{ if } x < x^* \end{cases},$$

for $\theta \in \{L, H\}$,

$$\pi(x,L) = \begin{cases} \bar{\pi} \text{ if } x > x^{**} \\ 0 \text{ if } x < x^{**} \end{cases},$$

and $\pi(x,H) = 0$.

Consider any equilibrium where x^* and x^{**} are less than 1. Then there is a region near 1 where beliefs are drifting up, the bad quality firm is censoring and working, and the firm never leaves this region. So in this region $D(x_t) = \frac{k\bar{\pi}}{r+\lambda}$, the difference in the payoff for the high and low quality firms comes purely from the low quality firm having to pay the additional cost of censorship. This naturally implies the following stark result

Proposition 7 *When $(1+\frac{r}{\lambda})c > k\bar{\pi}$, there is a unique MPE.*[7] *In that MPE the firm never invests in quality.*

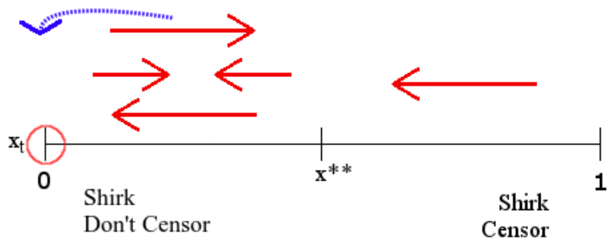

Figure 4: The structure of a full shirk equilibrium

The firm's ability to censor gives it quite a bit of control over how much it loses from quality switching from high to low. Since the firm can always censor bad news after quality switches, it never loses more than $k\bar{\pi}$ at each instant of time from being a low quality firm. Therefore it would never exert effort when $k\bar{\pi}$ is low enough relative to the cost of effort. If censoring bad news is too easy (for instance, cheaper than exerting effort), then the firm would always choose to substitute effort for censorship.

The intuition that leads to the firm never investing in quality holds for more general classes of equilibria than Markov Perfect. The argument that the firm never invests in quality straightforwardly generalizes to any equilibrium that uses calender time in addition to belief and firm type as the state variables, and in fact it holds for the class of all almost perfect bayesian equilibria.

[7] In this proposition and the subsequent proposition, uniqueness is taken to mean that all equilibria induce the same payoffs, distribution over beliefs and strategies are the same (except on a set of measure 0) except when the firm is indifferent between censoring and not. Since for some parameter values, beliefs may drift down to a point where the firm is indifferent, and then stay there until news arrives, the firm can play any strategy at these points as long as it induces the same drift of beliefs at these points.

Proposition 8 *If $(1+\frac{r}{\lambda})c > k\bar{\pi}$, the MPE from proposition 7 is the unique perfect bayesian equilibrium.*

This contrasts dramatically from the result in Board and Meyer-ter-Vehn (2013), who find that without the ability to censor bad news, there are equilibrium where the firm works. In fact, when $\lambda \geq \bar{\pi}$ and $\frac{\lambda\bar{\pi}}{(r+\lambda)(r+\bar{\pi})} > c$, there are constants $a < b$ such that all $x^* \in [a,b]$ are is a cutoff of an equilbria where the firm works above x^* if the firm was unable to censor. But giving the firm the ability to censor for a low enough cost completely destroys the firm's incentives to invest.

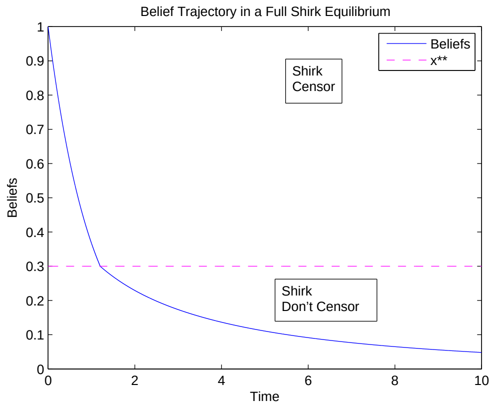

Figure 5: An example of the trajectory beliefs follow until news arrives in a full shirk equilibrium.

This full shirk equilibrium has many undesirable properties from the firm's perspective. While an equilibrium where the firm is working can have persistent reputation building, with probability 1 in a full shirk equilibrium the firm's reputation vanishes in finite time. A firm that could commit to not censoring would not suffer from these sorts of incentive problems. Board and Meyer-ter-Vehn (2013) identify the possibility of equilibria where the firm works when the bad news technology is exogeneous, but introducing sufficiently cheap censorship technology completely destroys these incentives.

Corollary 3 *Suppose $(1+\frac{r}{\lambda})c > k\bar{\pi}$. Let $(r_n)_{n=0}^{\infty}$ be a sequence of discount rates such that $r_n \to 0$. Consider a sequence of equilibrium with discount rate r_n, and a sequence of equilibria of the game where the firm has committed to not censor. Then:*

1. $\lim_{n\to\infty} r_n \left(V_n(x,\theta) + E\left(\int e^{-r_n t} k\pi_t dt | x_0 = x, \theta_0 = \theta\right)\right) \leq \liminf r_n V_{c,n}(x,\theta)$.
2. *This inequality is strict for any* $x > \min(1-\frac{\lambda}{\bar{\pi}}, \limsup x^*_{c,n})$[8]

The firm's willingness to pay for not censoring is always higher than the amount the firm expects to spend on censorship in the game with endogenous censorship as the firm becomes patient whenever the firm's reputation starts high enough. As the firm becomes patient, all that matters is where beliefs end up. In the game with censorship, 0 is the

[8] A sufficient condition for $x > \limsup x^*_{c,n}$ is if $\int_0^\infty e^{-\lambda t}\bar{\pi}\int_t^\infty e^{-\bar{\pi}s} x_s\, ds\, dt \geq \frac{c}{\lambda}$, where $x_0 = x$ and $\dot{x}_t = -\lambda x_t + \bar{\pi} x_t(1-x_t)$.

unique absorbing state, beliefs always eventually converge to 0. On the other hand, in the game where the firm has committed, in any region where beliefs are drifting up, with positive probability the firm eventually starts selling a high quality product, never stops, and beliefs drift up towards 1. So the firm receives a positive payoff with positive probability forever.

When $k\bar{\pi} > \left(1 + \frac{r}{\lambda}\right) c$, there may exist equilibria where the firm invests in quality. The firm now finds investing desirable because once it has a high quality product, it has either shut down bad news or no longer needs to censor bad news. There always exist equilibria of the form illustrated in the following figure.

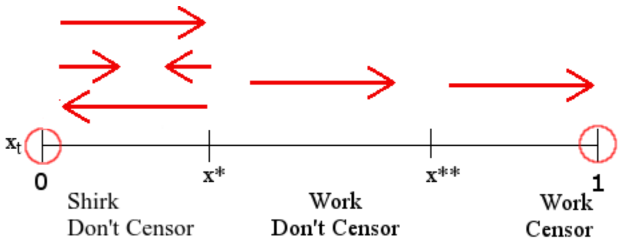

Figure 6: The structure of an equilibrium where the firm stops censoring before it stops working.

In this equilibrium, the firm works and censors when it has a high reputation, stops censoring at an intermediate reputation and stops working when its reputation is low enough. In equilibria of this type, as long as $x^* > 0$, beliefs can converge to 0 or 1. At 1, the firm has the incentive to invest because if the firm chooses not to invest and its product becomes a low quality product, it will then have to censor news to maintain its high reputation. But, since censoring is sufficiently expensive, the firm always has the incentive to invest in quality in order to not have to censor bad news. At 0, if $x^* > 0$, the firm can never convince consumers that it is investing in quality, so it never benefits from investing in quality, and its reputation can never recover. For very low c, the firm can have the incentive to work everywhere, in which case beliefs converge to 1 with probability 1.

In this equilibrium, beliefs depend crucially on the both the initial condition and are path dependent. In the region where the firm is censoring and investing, beliefs converge to 1 with probability 1. If beliefs start in the region near 0 where beliefs are drifting down, beliefs converge to 0 with probability 1. In every other region, with positive probability beliefs converge to either 1 or 0.

While the game without censorship from Board and Meyer-ter-Vehn (2013) has similar qualitative features, now a low quality firm with a high reputation can benefit from the ability to censor. The low quality firm can't be caught selling a low quality product if it's censoring, so it is better able to maintain a high reputation. On the other hand, a firm with a high quality product and a high reputation is only hurt by the ability to censor. Since the firm always has the incentive to invest in quality in the region where it has chosen to censor, a firm that starts with a high quality product never actually censors any bad news. On the other hand, the firm faces beliefs that drift up more slowly than they would if the firm could convince consumers that it was working and never censoring.

Equilibrium could also have the structure illustrated in the following figure.

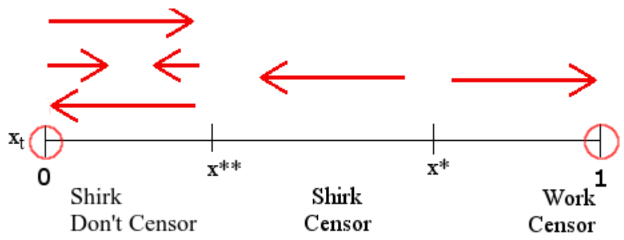

Figure 7: The structure of an equilibrium where the firm stops working before it stops censoring.

Equilibria with this structure may not always exist, but there are parameter values where equilibria of this form do exist. In this class of equilibria the limiting distribution of beliefs is entirely determined by the initial belief. Unless beliefs start in the region where the firm invests and censors, beliefs go to 0 with probability 1, and otherwise beliefs go to 1 with probability 1. As in the previous case, the ability to censor allows a low quality firm with a high reputation to preserve its reputation until it successfully develops a better product. On the other hand, it now allows a low quality firm with an intermediate reputation that never plans on investing in quality to prevent its reputation from dropping to 0 until it is already very low.

5 Exogenous News

In this section I consider the case of exogenous news. In the good news case, exogenous news means that good news can arrive even when the firm is not promoting, and in the bad news case, the firm can no longer perfectly censor negative information.

In the good news case, news now arrives at rate $(\mu + \pi_t)1_{\theta_t=H}$. Even when the firm is not promoting, good news still can arrive if the firm is selling a high quality product. In this case, beliefs follow the law of motion

$$\dot{x}_t = \lambda(\tilde{A}_t - x_t) - (\tilde{\Pi}_t + \mu_t)x_t(1 - x_t).$$

Even if the firm is not promoting, consumers still expect to see news and make inference based on that. As in the good news case with only endogenous news, the MPE can be expressed in terms of cutoffs.

Proposition 9 *Every Markov Perfect Equilibrium is characterized by two cutoffs $x^* \in [0, 1)$, $x^{**} \in (0, 1)$ where*

$$a(x) = \begin{cases} 1 \text{ if } x < x^* \\ 0 \text{ if } x > x^* \end{cases},$$

$$\pi(x, H) = \begin{cases} \bar{\pi} \text{ if } x < x^{**} \\ 0 \text{ if } x > x^{**} \end{cases},$$

and $\pi(x, L) = 0$.

1. *If* $\lambda \geq \mu$, *and* $\left(1 + \frac{r}{\lambda}\right) c \geq k\mu$ *then* $x^{**} \geq x^*$.
2. *If* $\lambda \geq \mu$, *and* $\left(1 + \frac{r}{\lambda}\right) c < k\mu$, *then* $x^* > x^{**}$ *or* $x^* = x^{**} = 0$.[9]

Without exogenous news the promotion cutoff was always above the investment cutoff, $x^{**} \geq x^*$, because if the firm was believed to be working but not promoting then consumers would have no way of detecting deviations. With exogenous news, this logic no longer holds. But the value of investing in quality is still determined by how much the firm expects to benefit from consumers finding out that it is selling a high quality product,

$$D(x) = \int_0^\infty e^{-(r+\lambda)t} [\mu\Delta(x_t) + \pi(x_t, H)(\Delta(x_t) - k)\, dt.$$

If the cost of investment is high, as in the first case, then $\Delta(x^*)$ needs to be relatively high to provide incentives to invest, which implies the firm also has incentive to promote.

In the first case, the firm's strategies have the same structure as they had in the perfect good news case. If promotion is sufficiently cheap, investment is sufficiently expensive or exogenous news is sufficiently inexpensive, the equilibrium structure remains the same. At a high reputation, the firm doesn't invest in quality or promote its product, at intermediate reputations the firm promotes high quality products, but does not invest, and at low reputations the firm invests and promotes.

In the long run, in these equilibria, the firm still uses promotion to manage its reputation. Investment is sufficiently expensive and exogenous news arrives sufficiently slowly that the firm wants to promote in order to try to delay when it has to start investing in quality. This is not the case when the cost of investment is sufficiently cheap or exogenous news is sufficiently effective , as in the second case.

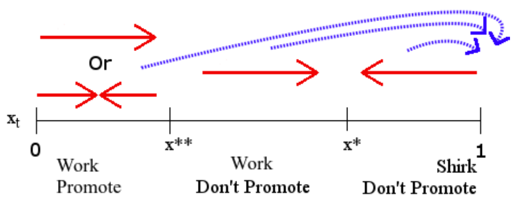

Figure 8: MPE in the second case.

In the second case, the firm finds it optimal to start investing in quality at a higher reputation than they start investing in promotion in order to take advantage of this exogenous news. Since $\lambda \geq \mu$, beliefs drift up when the firm is investing in quality and not promoting since quality shocks arrive faster than exogenous news. In turn, this means that start above the promotion cutoff will never drift below it, as illustrated in the figure.

[9]As before, existence is a bit of an issue here. If $\lambda \geq \mu + \bar{\pi}$, existence can be established using the argument from proposition 1 In the 2nd case, existence can be established whenever $\lambda \geq \mu$. x^* must be exactly the cutoff from an equilibrium in Board and Meyer-ter-Vehn (2013), and taking this cutoff fixed, a unique x^{**} can be pinned down.

Corollary 4 *Suppose $\lambda \geq \mu$, and $\left(1 + \frac{r}{\lambda}\right) c < k\mu$. Let τ_0 be the first time good news arrives. For any $t > \tau_0$, $\pi_t = 0$.*

Once a firm has established a high reputation, exogenous news is effective enough to credibly support investment in the absence of promotion. When the firm's reputation starts off sufficiently low, they promote until their reputation becomes sufficiently high, and then stop promoting. This is reminiscent of expensive restaurants, which may initially advertise or hold lavish grand openings, but eventually no longer need to spend effort on promotion because word of mouth and good reviews ensure that their reputation stays high. These established restaurants know that when they serve a great meal, people are going to find out, even if they don't do anything to advertise that fact.

Unsurprisingly, the bad news results are less clear. Consider a model where the firm can censor, but can't censor perfectly. In particular, bad news arrives at rate $\mu > \bar{\pi}$. Even without exogenous news, the equilibrium structure is ambiguous, there may be both equilibria where the firm stops censoring at a higher reputation than they stop working and equilibria where the reverse happens. It seems unlikely that adding exogenous news would clarify this ambiguity. I can check the robustness of the result that the firm never invests.

Proposition 10 *For any $\epsilon > 0$, $\mu - \bar{\pi} < \epsilon$ and $k\bar{\pi} \leq (1 + r/\lambda)c - \epsilon$, then there is a unique MPE in which the firm never invests in quality.*

The bad news results are robust to imperfect censorship, as long as the rate of bad news that cannot be censored is sufficiently small. If the firm can almost perfectly censor, then the firm still has very tight control over how much they benefit from having a high quality product relative to a low quality product. Censorship must be very expensive in order to for the firm to have incentives to credibly invest. Analogous results holds, if there is a small probability of exogenous good news, or if the firm can promote but can only induce a very low maximal arrival rate of good news. For space, I've omitted these results, but I'm happy to provide them upon request.

6 Conclusion

A firm's ability to control what information consumers see about a product is a crucial part of a firm's ability to build and maintain a reputation. Even as effort becomes arbitrarily cheap, a firm will still spend some time not working, because it knows there is no benefit from investment in quality unless it invests in promotion.

Censoring bad news can have much starker effects on incentives. While promotion and effort are complements, I exert effort because I know I can promote my successes in the future, effort and censorship are substitutes. This leads to cases where, when the cost of censorship is sufficiently lower, the firm never invests in its product, instead choosing to invest in hiding bad news about the product.

This is reminiscent of oil companies suppressing research about climate change or cigarette companies suppressing research about how unhealthy cigarettes are. While these companies could be investing in products that are safer and more effective, instead they've invested significant amounts of money in hiding bad news about their products. This has not only been bad for consumers, but seems to have also been bad for the companies, which are now trying to escape their negative reputations. Elaborate rebranding campaigns by Phillip Morris or British Petroleum seem to be designed entirely to escape the firm's reputation for selling dangerous, harmful products.

On the other hand, these properties may not extend to arbitrary public perfect equilibria that are not markovian in the public belief. As in Marinovic et al. (2015) and Halac and Prat (2014), there may be equilibria in richer state spaces that have different properties and rely on more complicated cycles of rewards and punishments through consumer's public beliefs. Equilibria that are markovian in the public belief capture features of how firm incentives for investment and signaling are driven by reputation concerns, but there certainly may be a larger class of equilibria that also have interesting features.

Controlling information is an important component of how a firm builds and maintains its reputation. The ability to promote and the ability to censor lead to interesting equilibrium dynamics and has real consequences on both firm strategies and firm payoffs. There are many other environments where a similar analysis may be worth investigating. For instance, it may be interesting to investigate how firm's can compete through the release of information. There are also many other ways a firm could control information; for instance, by hiding some of the history or choosing which summary statistics to display to consumers that may also be worthwhile to investigate.

References

Bar-Isaac, Heski and Steven Tadelis (2008), "Who wants a reputation." *Foundation and Trends in Microeconomics*.

Benabou, Roland and Guy Laroque (1992), "Using privileged information to manipulate markets: Insiders, gurus, and credibility." *The Quarterly Journal of Economics*, 921–958.

Board, Simon and Moritz Meyer-ter-Vehn (2013), "Reputation for quality." *Econometrica*, 81, 2381–2462.

Board, Simon and Moritz Meyer-ter-Vehn (2015), "A reputational theory of firm dynamics."

Bohren, Aislinn (2012), "Stochastic games in continuous time: Persistent actions in long-run relationships."

Bonatti, Alessandro and Johannes Hörner (2014), "Career concerns with exponential learning."

Che, Yeon-Koo and Johannes Hörner (2013), "Optimal design for social learning." Technical report.

Cisternas, Gonzalo (2015), "Two-sided learning and moral hazard." Technical report, Discussion paper, MIT Sloan.

Cripps, Martin W, George J Mailath, and Larry Samuelson (2004), "Imperfect monitoring and impermanent reputations." *Econometrica*, 72, 407–432.

Davis, Mark HA (1993), *Markov Models & Optimization*, volume 49. CRC Press.

Dellarocas, Chrysanthos (2005), "Reputation mechanism design in online trading environments with pure moral hazard." *Information Systems Research*, 16, 209–230.

Dilmé, Francesc (2014), "Reputation building through costly adjustment."

Ekmekci, Mehmet (2011), "Sustainable reputations with rating systems." *Journal of Economic Theory*, 146, 479–503.

Faingold, Eduardo and Yuliy Sannikov (2011), "Reputation in continuous-time games." *Econometrica*, 79, 773–876.

Fudenberg, Drew and David K Levine (1989), "Reputation and equilibrium selection in games with a patient player." *Econometrica: Journal of the Econometric Society*, 759–778.

Fudenberg, Drew and Jean Tirole (1986), "A" signal-jamming" theory of predation." *The RAND Journal of Economics*, 366–376.

Halac, Marina and Andrea Prat (2014), "Managerial attention and worker engagement."

Holmström, Bengt (1999), "Managerial incentive problems: A dynamic perspective." *The Review of Economic Studies*, 66, 169–182.

Holmstrom, Bengt and Paul Milgrom (1991), "Multitask principal-agent analyses: Incentive contracts, asset ownership, and job design." *Journal of Law, Economics, & Organization*, 24–52.

Hörner, Johannes and Nicolas Lambert (2015), "Motivational ratings."

Hu, Ju (2014), "Reputation in the presence of noisy exogenous learning." *Journal of Economic Theory*, 153, 64–73.

Hu, Ju (2015), "Biased learning and permanent reputation."

Keller, Godfrey and Sven Rady (2010), "Strategic experimentation with poisson bandits." *Theoretical Economics*, 5, 275–311.

Klein, Nicolas (2015), "The importance of being honest." *Theoretical Economics*.

Klein, Nicolas and Tymofiy Mylovanov (2010), "Expert experimentation." Technical report, working paper, University of Munich and Pennsylvania State University.

Klein, Nicolas and Sven Rady (2011), "Negatively correlated bandits." *The Review of Economic Studies*.

Kreps, David M, Paul Milgrom, John Roberts, and Robert Wilson (1982), "Rational cooperation in the finitely repeated prisoners' dilemma." *Journal of Economic Theory*, 27, 245–252.

Liu, Qingmin (2011), "Information acquisition and reputation dynamics." *The Review of Economic Studies*, 78, 1400–1425.

Liu, Qingmin and Andrzej Skrzypacz (2014), "Limited records and reputation bubbles." *Journal of Economic Theory*, 151, 2–29.

Mailath, George J and Larry Samuelson (2001), "Who wants a good reputation?" *The Review of Economic Studies*, 68, 415–441.

Mailath, George J and Larry Samuelson (2006), *Repeated games and reputations*, volume 2. Oxford university press Oxford.

Marinovic, Iván, Andrzej Skrzypacz, and Felipe Varas (2015), "Dynamic certification and reputation for quality." *Available at SSRN 2697762*.

Milgrom, Paul and John Roberts (1986), "Price and advertising signals of product quality." *The Journal of Political Economy*, 796–821.

Nelson, Phillip (1974), "Advertising as information." *Journal of political economy*, 82, 729–754.

Pei, Harry Di (2015), "Reputation with strategic information disclosure."

Tadelis, Steven (1999), "What's in a name? reputation as a tradeable asset." *The American Economic Review*, 89, 548.

Wiseman, Thomas (2009), "Reputation and exogenous private learning." *Journal of Economic Theory*, 144, 1352–1357.

Appendix 1 - Good News Proofs

Proof of Lemmas 1 and 2

These proofs are similar to the corresponding arguments in Board and Meyer-ter-Vehn (2013).

Lemma 1 Proof. These conditions follow from Lemma 5 of Board and Meyer-ter-Vehn (2013). They show that the function $\psi(t) = \int_t^\infty e^{-\int \rho(s)}\phi(t)dt$ is the unique bounded solution to

$$f(t) = \int_t^\infty \phi(s) - \rho(s)f(s)ds.$$

So the value function can be rewritten as

$$V(x,\theta) = \max_{a,\pi} \int_0^\infty x_t + A(x_t,\theta)(\lambda D(x_t) - c) + \Pi(x_t,\theta)1_{\theta=H}(\Delta(x_t) - k) + \lambda V(x_t, L) - \lambda V(x_t,\theta) - rV(x_t,\theta)dt$$

where x_t solves $\dot{x}_t = (\tilde{A}_t - x_t)\lambda - \tilde{\Pi}_t x_t(1-x_t)$ with $x_0 = x$.

Therefore the optimal $a(x_t,\omega)$ solves

$$\max_{a\in[0,1]} \lambda D(x_t)a - ca$$

and the optimal $\pi(x_t, H)$ solves

$$\max_{\pi\in[0,\bar{\pi}]} \Delta(x_t)\pi - k\pi.$$

since these maximize the integrand pointwise. ■

Lemma 2 Proof. Subtracting the two value functions gives

$$D(x) = \int_0^\infty \pi(x_t, H)(\Delta(x_t) - k) - (r+\lambda)D(x_t)dt.$$

Using Lemma 5 of Board and Meyer-ter-Vehn (2013) again,

$$D(x) = \int_0^\infty e^{-(r+\lambda)t}\pi(x_t, H)(\Delta(x_t) - k)dt.$$

Lemma 5 *If $x_0 > x_0^*$ then $x_t \geq x_t^*$ for all $t < \tau$, where τ is the first time news arrives.*

Proof. Both x_t and x_t^* are continuous in t. If $x_t = x_t^*$ at any t, then for any $s > t$, x_s and x_s^* both solve

$$\dot{x}_s = \lambda(a(x_s) - x_s) - \pi(x_s, H)x_s(1-x_s).$$

with the same initial condition. Since the solution to this is the unique solution that is consistent with the discrete time approximation, x_s^* and x_s must be equal. So if x_t and x_t^* cross, they must be the same from then on, so $x_t \geq x_t^*$. ■

This implies that the value functions are increasing in x. For any two initial conditions, $x > x'$, the firm facing a sequence of consumers with prior $x_0 = x$ could instead follow the strategy it would have followed if it faced a sequence of consumers with prior $x_0 = x'$. This would induce the same probability measure over signals, quality shocks, and quality. Moreover, by Lemma 5, the flow payoffs would be strictly higher than they were for the firm that starts at x'. Since equilibrium payoffs must be (weakly) greater than this, value functions must be increasing. Therefore, $V(x, \omega)$ is increasing in x.

The value of a signal $\Delta(x) = V(1, H) - V(x, H)$ is decreasing since $V(x, H)$ is increasing.

The high quality firm gets a higher payoff than the low quality firm since it can generate the exact same distribution over signals and quality as a firm that starts with low quality. So $V(x, H) \geq V(x, L)$.

Finally, $D(x)$ is decreasing. This difference can be written as

$$\int_0^\infty \pi(x_t, H)[\Delta(x_t) - k]dt,$$

where $\pi(x_t, H)$ maximizes

$$\pi(x_t, H)[\Delta(x_t) - k].$$

For larger x_t, $\Delta(x_t) - k$ is smaller. Therefore, if beliefs start at a larger x_0, the integrand is smaller pointwise. So $D(x)$ is decreasing. ■

Proposition 1 - Existence and Structure

The result that the equilibrium can be characterized in terms of two cutoffs follows directly from the monotonicity properties. Existence of equilibrium and $x^{**} \geq x^*$ are slightly more involved.

Lemma 6 *In any equilibrium, there exists points x^{**} and x^* such that*

$$a(x) = \begin{cases} 1 \text{ if } x < x^* \\ 0 \text{ if } x > x^* \end{cases},$$

$$\pi(x, H) = \begin{cases} \bar{\pi} \text{ if } x < x^{**} \\ 0 \text{ if } x > x^{**} \end{cases}.$$

Moreover, $x^{**} \geq x^*$.

Proof. Consider any equilibrium, let x^* be the highest public belief where the firm invests in quality[10] and let x^{**} be the highest public belief where the firm promotes.

[10]It follows from this proposition that it is also the lowest x where the firm doesn't invest

Suppose, that $x^* > x^{**}$. Then consider any belief $x' \in (x^*, x^{**})$ where the firm invests. There are a few possibilities that must be considered.

If beliefs are drifting up or are constant at this point, then they never cross x^*. If the firm followed this strategy there would be no arrivals of news and the firm would receive

$$\int e^{-rt}[x_t - ca_t]dt.$$

Therefore the firm prefers to never work, since it gets a payoff of $\int_0^\infty e^{-rt}x_t$ from never working.

It remains to to consider the case where beliefs are drifting down at all points in (x^*, x^{**}). Since beliefs would be drifting up if $a(x') = 1$, and the firm is assume to be investing, it must be that $a(x') \in (0, 1)$. This can only happen if $D(x) = \frac{c}{\lambda}$, for all $x' - \epsilon < x < x'$ and some $\epsilon > 0$, since the firm needs to choose a level of investment in $(0, x)$ for beliefs to drift down. This means that $D(x) = \frac{c}{\lambda}$ for all $x \in (x^*, x^{**})$. But

$$D(x') = \int_t^{t+\delta} e^{-(r+\lambda)}0\, dt + e^{-(r+\lambda)\delta}D(x)$$

for sufficiently small δ and some x, s.t.$x' - \epsilon < x < x'$. So $D(x') < \frac{c}{\lambda}$, but the firm was assumed to be exerting effort at that point, which is a contradiction.

Therefore, $x^{**} \geq x^*$. Moreover, $D(x)$ cannot be constant below x^* because

$$\pi(x_t, H)(\Delta(x_t) - k) = \bar{\pi}(\Delta(x_t) - k),$$

so for any two beliefs x, x' below x^*, $D(x_0) > D(x'_0)$ if $x'_0 > x_0$ since the integrand at any point x'_t is strictly less than the integrand at x_t. ■

A MPE Exists

Proposition 11 *A Markov Perfect Equilibrium Exists.*

Proof. Let $U_\theta : [0, 1]^3 \to \mathbb{R}$ as

$$U_\theta(x_0, x^*, x^{**}) = \max_{a,\pi} E_{a,\pi}\left(\int e^{-rt}[x_t(x^*, x^{**}) - ca_t - k\pi_t]dt|\theta_0 = \theta, x_0 = x\right)$$

where $x_t(\cdot)$ behaves as follows. The firm is believed to be playing according to cutoffs x^* and x^{**}. So the firm works below x^* and promotes below x^{**}. Belief follow the modified the law of motion induced by Bayes rule and are consistent with admissibility[11].

[11] Specifically when the drift switches signs so that its positive below the cutoff and negative above the cutoff, it is 0 at the cutoff, at any cutoff where the drift doesn't change sign, the choice of strategy doesn't matter.

Let

$$\Gamma(x^*, x^{**}) = \begin{cases} \arg\min_{\hat{x}^* \in [0,1]} \int_{\hat{x}^*}^1 \lambda(U_H(s, x^*, x^{**}) - U_L(s, x^*, x^{**})) - c\, ds \\ \arg\min_{\hat{x}^{**} \in [0,1]} \int_{\hat{x}^{**}}^1 U_H(1, x^*, x^{**}) - U_H(s, x^*, x^{**}) - k\, ds \end{cases}.$$

Since the value functions are bounded, the objective functions are continuous and well defined, so this operator is well defined and non-empty. By the monotonicity of $D(s, x^*, x^{**}) = U_H(s, x^*, x^{**}) - U_L(s, x^*, x^{**})$ and $\Delta(s, x^*, x^{**}) = U_H(1, x^*, x^{**}) - U_H(s, x^*, x^{**})$ this is convex valued.

Continuity in Initial Values

Lemma 7 *For given cutoffs (x^{**}, x^*) and for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any $\hat{x}_0 \in [x_0 - \delta, x_0 + \delta]$ and any history then*

$$\int e^{-rt} |x_t(x^*, x^{**}) - \hat{x}_t(x^*, x^{**})| dt \leq \epsilon.$$

Lemma 8 *$U_\theta(\cdot, x^*, x^{**})$ is continuous in the first argument.*

Lemma 7 Proof. After the first arrival, beliefs coincide forever, so it is sufficient to show that $\hat{x}_t$ and x_t stay close together before an arrival. For some cases, it may be easier to consider the log likelihood ratio $l_t = \log(x_t/(1 - x_t))$. Fix $l_0 > \hat{l}_0 > l_0 - \delta$, the argument is analogous for the other case.

1. l_0 and $\hat{l}_0$ are in the work/no promote region.

   l_t and $\hat{l}_t$ are both drifting at up at rate $\lambda(1 + e^{-l})$, which is decreasing in l, so these beliefs are getting closer together. Since the drift at $x^* = 0$, beliefs never leave this region. Therefore, as long as $|l_0 - \hat{l}_0| < r\epsilon$ then $|l_t - \hat{l}_t| < r\epsilon$ for all t.

2. l_0 is in the work/no promote region, $\hat{l}_0$ is in the work/promote region.

   This implies that $x^* > x^{**}$.

   l_t and $\hat{l}_t$ are both drifting up. $\hat{l}_t$ drifts up at least at rate $\lambda - \bar{\pi}$, so it reaches x^{**} in finite time. Then beliefs are in the situation from 1. So if $|l_0 - \hat{l}_0| < \delta$, the time it takes for l to reach l^{**} is at most $\delta/(\lambda - \bar{\pi})$. In that amount of time, $\hat{l}_t$ is at most drifting up at rate λ, so $|l_t - \hat{l}_t| < \frac{\lambda\delta}{\lambda - \bar{\pi}}$ for any time $t \leq \delta/(\lambda - \bar{\pi})$.

   Then, as long as $\frac{\lambda\delta}{\lambda - \bar{\pi}} < r\epsilon$, by the argument from 1. for any $t > \frac{\delta}{\lambda - \bar{\pi}}$, $|l_t - \hat{l}_t| < r\epsilon$.

3. l_0 is in the shirk/promote region or the shirk/no promote and $\hat{l}_0$ is in the work/promote region.

   x^* is convergent, the drift above the cutoff is negative and the drift below the cutoff is positive, so beliefs are drifting together. As long as $|l_0 - \hat{l}_0| < r\epsilon$, then $|l_t - \hat{l}_t| < r\epsilon$.

4. l_0 and $\hat{l}_t = 0$ are both in the work/promote region.

   The reputational drift $\lambda(1 + e^{-l}) - \bar{\pi}$ is decreasing in the work/promote region, so beliefs are drifting closer together as long as they both lie in that region. If

$x^* < x^{**}$, then beliefs never leave this region and otherwise they eventually are in the situation from case 2. In either case, there exists a sufficiently small δ such that if $|l_0 - \hat{l}_0| < \delta$ then $|l_t - \hat{l}_t| < r\epsilon$ for all t.

5. l_t is in the shirk/no promote region, $\hat{l}_t$ is in the work/no promote region. l_t is drifting down while $\hat{l}_t$ is drifting up, so as long as $|\hat{l}_0 - l_0| < r\epsilon$, then $|l_t - \hat{l}_t| < r\epsilon$.

6. l_t and $\hat{l}_t$ are in the shirk/promote region.

   In this region, x_t and $\hat{x}_t$ are determined by a standard differential equation of the form $x'(t) = f(x,t)$ and $x_0 = x$, $f(x,t)$ is a continuous function and is lipschitz in x, so by standard results from the differential equation literature, it is continuous in the initial conditions until $\hat{x}$ leaves the shirk/promote region. If the shirk/promote cutoff is interior, δ can be chosen so that beliefs are at most δ' apart when they hit that cutoff, in which case an argument from a previous case applies. Otherwise δ can be chosen so that beliefs are at most δ' apart when $\hat{x}_t$ hits 1/2, after which, the x_t and $\hat{x}_t$ only drift apart a small amount more before x_t hits 1/2 and they start drifting together since $\dot{x}_t - \dot{\hat{x}}_t = \lambda(\hat{x}_t - x_t) + \bar{\pi}(\hat{x}_t(1-\hat{x}_t) - x_t(1-x_t)) < 0$ below 1/2.

7. l_t is in the shirk/no promote region, $\hat{l}_t$ or the shirk/promote region. l_t is drifting down at least at rate λ, so it enters the shirk/promote region in at most time δ/λ. So, the most beliefs can seperate before l_t enters the shirk/promote region, after which we are in the previous case. So δ can be choosen so that these beliefs never drift far apart.

8. l_t and $\hat{l}_t$ are in the shirk/no promote region.

   In this region $\dot{x}_t - \dot{\hat{x}}_t = \lambda(\hat{x}_t - x_t) < 0$, so beliefs are drifting close together until they enter a region from one of the other cases.

Finally, bounding the distance between the likelihood ratios is sufficient because, by the mean value theorem

$$|x_t - \hat{x}_t| \leq \max_x \frac{1}{(\frac{d}{dx}\log x/(1-x))}|l_t - \hat{l}_t| \leq |l_t - \hat{l}_t|.$$

■

Lemma 8 Proof. This follows directly from the previous claim. The optimal strategy is independent of the initial conditions, and x_t is continuous in the initial condition, so $U_\theta(\cdot, x^*, x^{**})$ must be continuous. ■

Continuity in Cutoffs

Lemma 9 *$U_\theta^\gamma(x_0, x^*, x^{**})$ is continuous in x^* and x^{**}.*

Proof. It is sufficient to show that for cutoffs (x^*, x^{**}) there is a $\delta > 0$ s.t. $(\hat{x}^*, \hat{x}^{**}) \in [x^* - \delta, x^* + \delta] \times [x^{**} - \delta, x^* + \delta]$ such that

$$E_{a,\pi}\left(\int e^{-rt}[x_t(x^*, x^{**}) - \hat{x}_t(\hat{x}^*, \hat{x}^{**})|dt\right) \leq \epsilon.$$

since

$$\begin{aligned} U_\theta(x_0, x^*, x^{**}) &\geq E_{\hat{a},\hat{\pi}}\left(\int e^{-rt}[x_t - ca_t - k\pi_t]dt|\theta_0 = \theta, x_0 = x\right) \\ &\geq E_{a,\pi}\left(\int e^{-rt}[\hat{x}_t - c\hat{a}_t - k\hat{\pi}_t]dt|\theta_0 = \theta, x_0 = x\right) - \epsilon \geq U_\theta(x_0, \hat{x}^*, \hat{x}^{**}) \end{aligned}$$

where $\hat{a}, \hat{\pi}$ is the optimal strategy if the cutoffs are $\hat{x}^*$ and $\hat{x}^{**}$. Since the problem is symmetric, this implies that U_θ is continuous.

Continuity in x^*. Fix x^{**} and a strategy profile (a, π). I need to show that for any $\epsilon > 0$ there's a δ s.t. for $\hat{x}^* \in [x^* - \delta, x^* + \delta]$ then

$$\int e^{-rt}|x_t - \hat{x}_t|dt \leq \epsilon.$$

Its convenient to define some terms here. A cutoff X convergent if $\lim_{x \to X+} g(x) \geq 0$, $\lim_{x \to X-} g(x) \leq 0$, divergent if $\lim_{x \to X+} g(x) < 0$ and $\lim_{x \to X-} g(x) > 0$ and permeable if it is neither convergent or divergent. At a convergent cutoffs, beliefs that start in a neighborhood of the cutoff converge to it, and stay there until an arrival, at a divergent cutoff beliefs are pushed away from it, and at a permeable cutoff beliefs cross the cutoff and continue to drift in the same direction.

Any investment cutoff x^* is convergent. Since $\lambda \geq \bar{\pi}$, below x^* beliefs must be drifting up, and above it beliefs are drifting down. Therefore, x^* must be a convergent cutoff.

Let $\hat{x}^* \in [x^*, \min(x^* + r\epsilon, 1)]$. Then either $\hat{x}^*$ is convergent, or the convergent point for the $\hat{x}$ process is between $\hat{x^*}$ and x^*. The largest possible distance between the two processes is the distance between these two cutoffs, since beliefs coincide until they hit the cutoffs, and then get stuck at the two convergent points until an arrival. So $|\hat{x}_t - x_t| \leq r\epsilon$.

This argument can be reversed to show continuity in the other direction.

Continuity in x^{**}. Fix investment cutoff x^*. For any $\epsilon > 0$, there exists a $\delta > 0$ such $\hat{x}^{**} \in [x^{**}, (x^{**} + \delta)]$ then $|\hat{x}_t - x_t| < r\epsilon$.

x^{**} is a permeable cutoff. Let T be the time such that $\hat{x}_T = x^{**}$. For any x_0, beliefs drift at the same rate unless $x_0 > x^{**}$ and drift at the same rate until they enter the region between $\hat{x}^{**}$ and x^*. So it is sufficient to show that δ can be chosen so that the most beliefs can separate from initial condition $\hat{x^{**}}$ is bounded by $r\epsilon$.

After x_t crosses x^{**} beliefs follow the same law of motion, just with different initial conditions, so for any ϵ there exists a $\delta' > 0$ such that as long as $|\hat{x}_T - x_T| < \delta'$ then $|\hat{x}_t - x_t| < r\epsilon$ for all $t > T$.

If $x^{**} = 0$, then if $\delta < r\epsilon$ it must be that $|\hat{x}_t - x_t| < r\epsilon$. Otherwise, x_t drifts down at least at rate $-\lambda x^{**}$ while $\hat{x}_t$ drifts down by at most $-\lambda\hat{x}^{**} - \frac{1}{4}\bar{\pi}$ before time T. So, the furthest apart beliefs can drift before time T is $(\frac{1}{4}\bar{\pi} + \lambda(\hat{x}^{**} - x^{**}))T$. Moreover, $T < \delta/(\lambda x^{**})$. So, δ can be chosen so that for any x_0, $|\hat{x}_t - x_t| < r\epsilon$, and therefore $\int e^{-rt}|x_t - \hat{x}_t|dt \leq \epsilon$.

This argument can be reversed to show continuity in the other direction. ■

By the dominated convergence theorem, the objective function is continuous. Γ is UHC, convex valued (by monotonicity of $\Delta(\cdot)$ and $D(\cdot)$), compact valued and non-empty. So it has a fixed point. This fixed point satisfies the sequential rationality conditions for the law of motion induced by the fixed point since they are fixed points of

$$\Gamma(x^*, x^{**}) = \begin{cases} \arg\min_{\hat{x}^* \in [0,1]} \int_{\hat{x}^*}^1 \lambda(U_H(s, x^*, x^{**}) - U_L(s, x^*, x^{**})) - c\, ds \\ \arg\min_{\hat{x}^{**} \in [0,1]} \int_{\hat{x}^{**}}^1 U_H(1, x^*, x^{**}) - U_H(s, x^*, x^{**}) - k\, ds \end{cases},$$

and since value functions are monotone, the optimal cutoffs satisfy

$$U_H(s, x^*, x^{**}) - U_L(s, x^*, x^{**}) \geq (\leq) c/\lambda$$

below (above) the fixed point x^* and

$$U_H(1, x^*, x^{**}) - U_H(s, x^*, x^{**}) \geq (\leq) k$$

below (above) the fixed point x^{**}. So, this fixed point constitutes an equilibrium.

■

Propositions 4 - Limit Behavior

Proposition 4 *As $c \to 0$, there is a unique limit. In the limit as $c \to 0$, equilibrium payoffs are increasing in k as long as $\lambda \geq \bar{\pi}$.*

Lemma 10 *For any sequence $c_n \to 0$ and for any sequence of equilibria, $|x_n^* - x_n^{**}| \to 0$.*

Suppose not. Then there exists an $\epsilon > 0$ and a subsequence s.t. $|x_{n_k}^* - x_{n_k}^{**}| > \epsilon$ for all k. I'm going to drop the k subscript for convenience. Consider $\Delta_n(x_n^*)$. This can be rewritten as

$$\begin{aligned} \Delta_n(x_n^*) - k &= V_n(1, H) - k - V_n(x_n^*, H); \\ &= V_n(x_n^{**}, H) - V_n(x_n^*, H); \\ &> \int_{T^{**}}^{\infty} e^{-(r+\bar{\pi}+\lambda)t}[(x_{n,t} - x_n^*) + \lambda(V_n(x_{n,t}, L) - V_n(x_n^*, L))]dt; \\ &> \int_{T^{**}}^{\infty} e^{-(r+\bar{\pi}+\lambda)t}[(x_{n,t} - x_n^*)]dt. \end{aligned}$$

The first inequality comes from considering the deviation where after beliefs hit x^{**}, the firm plays the strategy it would have played at x^* and the second comes from the monotonicity of the value function.

From the firm's perspective, the worst possible law of motion for $x_{n,t}$ is bounded below by $-\lambda - \frac{1}{4}\bar{\pi}$ until $x_{n,t} = x^*$ and then $\dot{x}_{n,t} = 0$ after that. This gives

$$\Delta(x^*) - k > \int_0^{\frac{\epsilon}{\lambda + \frac{1}{4}\bar{\pi}}} e^{-(r+\bar{\pi}+\lambda)}(\epsilon - (\lambda + \frac{1}{4}\bar{\pi})u)du = B.$$

But, as $n \to \infty$, $c \to 0$. Moreover,

$$D(x_n^*) > \int e^{-(r+\lambda)t}[B]dt = \frac{1}{r+\lambda}B$$

But, $\lambda D(x_n^*) \leq c_n$ for all n by sequential rationality and $c_n \to 0$. This is a contradiction. So $|x_n^* - x_n^{**}|$ must converge.

Lemma 11 *The cutoff x_n^{**} converges to a unique cutoff. Cutoff determined by solution to:*

$$rk = \frac{r}{r+\lambda} + \frac{\lambda}{r+\lambda}e^{-(r+\lambda)T} - e^{-\lambda T},$$

where $x^{**} = e^{-\lambda T}$.

Suppose that the sequence of x_n^{**}'s did not converge to the cutoff x^{**}. Let T_n^* be the amount of time it takes for beliefs to go from 1 to x_n^*. Then there exists an $\epsilon > 0$ s.t.

$$\left| rk - \left(\frac{r}{r+\lambda}(1 - e^{-(r+\lambda)T_n^*}) + e^{-(r+\lambda)T_n^*} - e^{-\lambda T_n^*} \right) \right| \geq \epsilon$$

infinitely often. The value function can be rewritten as

$$\begin{aligned}
V_n(1, H) = &\int_0^{T_n^{**}} e^{-(r+\lambda)t} x_{n,t} + \lambda V_n(x_{n,t}, L)dt; \\
&+ \int_{T_n^{**}}^{T_n^*} e^{-(r+\lambda)t - \bar{\pi}(t - T_n^{**})}[x_{n,t} + \bar{\pi}(V_n(1, H) - k) + \lambda V_n(x_{n,t}, L)]dt \\
&+ e^{-(r+\bar{\pi}+\lambda)T_n^*} V_n(x^*, H); \\
= &\int_0^{T_n^{**}} e^{-rt} x_{n,t} dt + \int_{T_n^{**}}^{\infty} e^{-ru} x_{n,u}(1 - e^{-\lambda T_n^*}) \\
&+ \int_{T_n^{**}}^{T_n^*} e^{-(r+\bar{\pi}+\lambda)t}[x_{n,t} + \bar{\pi}(V_n(1, H) - k) + \lambda V_n(x_{n,t}, L)]dt \\
&+ (e^{-(r+\lambda)T_n^*} - 1)[x_n^* + c].
\end{aligned}$$

And using the indifference condition at the cutoff and the convergence of the two cutoffs,

$$
\begin{aligned}
V_n(1,H) - V_n(x^{**},H) &= \int_0^{T_n^{**}} e^{-rt} x_{n,t} dt + \int_{T_n^{**}}^{\infty} e^{-ru} x_{n,u}(1 - e^{-\lambda T_n^*}) \\
&+ \int_{T_n^{**}}^{T_n^*} e^{-(r+\bar{\pi}+\lambda)t}[x_{n,t} + \bar{\pi}(V_n(1,H) - k) + \lambda V_n(x_{n,t},L)]dt \\
&+ (e^{-(r+\lambda)T_n^*} - 1)[x_n^* + c] - V_n(x^{**},H) \\
\Delta(x^{**}) &= \int_0^{T_n^{**}} e^{-rt} x_{n,t} dt + \int_{T_n^{**}}^{\infty} e^{-ru} x_{n,u}(1 - e^{-\lambda T_n^*}) \\
&+ \int_{T_n^{**}}^{T_n^*} e^{-(r+\bar{\pi}+\lambda)t}[x_{n,t} + \bar{\pi}(V_n(1,H) - k) + \lambda V_n(x_{n,t},L)]dt \\
&+ (e^{-(r+\lambda)T_n^*} - 1)[x_n^* + c] \\
&- e^{(r+\bar{\pi}+\lambda)T_n^{**}}(\int_{T_n^{**}}^{T_n^*} e^{-(r+\bar{\pi}+\lambda)t}[x_{n,t} + \bar{\pi}(V_n(1,H) - k)] + \lambda V_n(x_{n,t},L)dt \\
&+ e^{-(r+\lambda-\bar{\pi})(T_n^* - T_n^{**})} V_n(x^*,H)) \\
rk &= \int_0^{T_n^*} re^{-(r)t} x_{n,t} dt + (1 - e^{-\lambda T_n^*}) \int_{T_n^*}^{\infty} re^{-rt} x_{n,t} dt + e^{-(r+\lambda)T_n^*} x_n^* - x_n^* + \delta(T_n^*, T_n^{**}) \\
rk &= \frac{r}{r+\lambda}(1 - e^{-(r+\lambda)T_n^*}) + (1 - e^{-\lambda T_n^*})e^{-(r+\lambda)T_n^*} + e^{-(r+\lambda)T_n^*} x_n^* - x_n^* + \delta(T_n^*, T_n^{**}) \\
rk &= \frac{r}{r+\lambda}(1 - e^{-(r+\lambda)T_n^*}) + e^{-(r+\lambda)T_n^*} - e^{-\lambda T_n^*} + \delta(T_n^*, T_n^{**})
\end{aligned}
$$

Where the second equation comes from expanding $V_n(x_t, L)$ out, and using integration by parts, and the integrals are finally evaluated by using the fact that before time T^*, beliefs are equal to $e^{-\lambda t}$.The term $\delta(T_n^*, T_n^{**})$ collects all the terms that are integrals from T_n^{**} to T_n^*, and goes to 0 as $T_n^* - T_n^{**} \to 0$. Since there exists an N such that for all $n \geq N$ $\delta(T_n^*, T_n^{**}) < \epsilon$, this is a contradiction.

It remains to consider what happens if x_n^{**} is 0 infinitely often. This can only happen if

$$k > \Delta_n(0) = \int_0^{\infty} e^{-(r+\lambda)t} dt.$$

Suppose the sequence (x_n^{**}) had another subsequence that was never 0. Then, by the logic from above, in the limit of this subsequence

$$rk = \frac{r}{r+\lambda}(1 - e^{-(r+\lambda)T^*}) + e^{-(r+\lambda)T^*} - e^{-\lambda T^*}.$$

But, the left hand side of this is bounded above by $\frac{r}{r+\lambda}$, and rk is bounded below by $\frac{r}{r+\lambda}$, so this cannot hold. Therefore, the sequence must not be non-zero infinitely often. So $x_n^{**} \to 0$, and the proposition is satisfied.

The properties of the limit equilibrium then follow directly.

Commitment in the limit

Proof of Proposition 5

As $c \to 0$, $V_{BM}(1,H) = V_{BM}(1,L) = \frac{1}{r} > V(1,H)$. Let $x_{BM,t}$ be beliefs in the commitment game. $x_{BM,t} \geq x_t$ until the first arrival (since $\lambda \geq \bar{\pi}$, and the firm that has committed is believed to be exerting effort everywhere), and after the first arrival $x_{BM,t} = 1$ forever, while x_t still drifts down. Therefore, even with costs added back in, the committed firm receives higher flow payoffs everywhere, and strictly higher payoffs after the first arrival, so

$$V_{BM}(x,\theta) > V(x,\theta) + E\left(\int_0^\infty e^{-rt} k\pi_t dt | x_0 = x, \theta_0 = \theta\right).$$

Commitment to a Promotion Strategy

Proof of Proposition 3

Proposition 12 *An optimal commitment cutoff strategy exists*

Proof. This will be implied by the following two lemmas

Lemma 12 *Given a promotion cutoff x^{**}, there is a unique equilibrium investment cutoff.*

Proof.

Consider two investment cutoffs, x^* and x'^*, $x^{**} > x^* > x'^*$ and suppose that both were consistant with an equilibrium. As before, let $D(x, x^*, x^{**})$ represent the difference between the two value functions if the believed cutoffs are x^* and x^{**} and $x_0 = x$. If both cutoffs were consistent with equilibrium, then

$$D(x^*, x^*, x^{**}) - D(x^*, x'^*, x^{**}) = E\left(\int_0^\infty e^{-rt} x_t - x'_t dt\right) - \int e^{-rt}(x_t - x'_t)dt < 0.$$

These beliefs drift apart the same amount until the firm successfully promotes, afterwhich the first set of beliefs are closer together and never spread farther apart then the beliefs conditional on no news arriving, so this difference is strictly negative. Therefore, since, under optimal play $D(x, x'^*, x^{**})$ is decreasing for every law of motion

$$D(x^*, x^*, x^{**}) < D(x'^*, x'^*, x^{**}),$$

Therefore the cutoff must be unique, the function $x^* \mapsto D_{x^{**},x^*}(x^*) = \frac{c}{\lambda}$ at most once, or is always less than c/λ. This cutoff $x^*(x^{**})$ is continuous in x^{**}, since $D_{x^{**},x^*}(x)$ is continuous. ■

Lemma 13 $x^{**} \mapsto V_c(x_0, \theta_0)$ *is continuous in* x^{**}.

Proof. For any $\epsilon > 0$, there exists a $\delta > 0$ such that if $x^{**} - x'^{*}* < \delta$. Let a' and π' denote the stratgies under the x' law of motion.

$$V(x_0, \theta_0, x^*(x^{**}), x^{**}) \geq E\left(\int_0^\infty e^{-rt}x_t - ca'_t - k\pi'_t\, dt\right) - \frac{\epsilon}{2} \geq V(x_0, \theta_0, x^*(x'^{**}), x'^{**}) - \epsilon$$

since, by argument from the existence proof, the law of motion varies continuously in the cutoffs, and the promotion cutoff can be made to move sufficiently small that it only adds a $\epsilon/2$ probability of an arrival, which in turn can increase the firm's value by at most 1. ■

Therefore the mapping $x^{**} \mapsto V_c(x_0, \theta_0)$ is continuous, the domain is a compact space, so a maximum exists. Therefore, an optimal promotion strategy exists. ■

Lemma 14 *Let $V_c(x, \theta)$ be the value function for a commitment game where the firm has committed to a promotion cutoff x_c^{**}. If $V_c(x, \theta) > V(x, \theta)$, then $x^{**} > x_c^{**}$.*

Proof. Note that in this new game, the sequential rationality condition for the investment decision still holds.

Let x^* and x^{**} be cutoffs of the equilibrium that induces the highest cutoff for investing in quality. Suppose that the firm deviates to a promotion cutoff above x^{**}. Let V_{dev} denote the value function for the game with the new promotion cutoff but the old invest/not invest cutoff where the firm has deviated to playing the strategy prescribed by x_c^*, x_c^{**}. Then $V_{dev}(1, H) < V(1, H)$ as is $V_{dev}(x^*, H) < V(x^*, H)$, because the firm is would have been playing suboptimally in the original game, and if beliefs follow the law of motion induced by this new cutoff, payoffs decrease further, because the same measure over can be induced, but beliefs drift down faster. Moreover, in this game with the new law of motion, the firm doesn't want to invest at any belief above x^*, since at x^*,

$$D(x^*) = \frac{1}{r + \lambda}[V(1, H) - \frac{x^* + \bar{\pi}V(1, H) - c - k\bar{\pi}}{\bar{\pi} + R} \leq \frac{c}{\lambda}$$

since the value at 1 must be strictly lower than the value in the original game where the firm was indifferent between investing and not at x^*. Since $D(\cdot)$ is decreasing, this implies the firm finds it optimal to not invest at any belief above x^*, and is at most indifferent at x^*.

In order for committing to this to be optimal, it must induce a higher level of investment in quality. If it did not, then deviating to follow the strategy in the new game would be a profitable deviation in the original game, since absent arrival beliefs are weakly higher at every point, and the firm can always deviate to induce the same measure over arrivals.

This means there must be some $x_c^* > x^*$ such that

$$D_c(x_c^*) = \int e^{-(r+\lambda)t}\bar{\pi}\left[V_c(1, H) - \frac{x_c^* - c + \bar{\pi}[V_c(1, H) - k]}{r + \bar{\pi}}\right] \geq c/\lambda$$

This did not hold before beliefs updated, since in equilibrium at x_c^*, $D(x_c^*) < c/\lambda$.

But, $D_c(x_c^*) < D_{dev}(x_c^*)$, the $D_{dev}(x)$ since

$$\begin{aligned} D_c(x_c^*) - D_{dev}(x_c^*) &= V_c(x_c^*, H) - V_{dev}(x_c^*, H) - [V_c(x_c^*, L) - V_{dev}(x_c^*, L)] \\ &= E(\int_0^\infty e^{-rt}[x_{c,t} - x_{dev,t}]) - \int e^{-rt}[x_{c,t} - x_{dev,t}] < 0 \end{aligned}$$

This is negative, because the two strategies induce the same measure over arrivals, which in turn implies that if the firm starts with a high quality product $V_c(x, H) - V_{dev}(x, H)$, the beliefs must alwasy be closer together (weakly), since they are either exactly the same distance apart independent of quality if no news arrives, and if news arrives they are equal until they hit x_c^*, in which case the low quality beliefs have already seperated more. But, with optimal play, the value function for a firm with low quality doesn't change from V_{dev}, while the value a firm with high quality gets increases, so under this new law of motion $D_c(x_c^*) < D_{dev}(x_c^*) \leq D(x_c^*) < \frac{c}{\lambda}$, which is a contradiction. ■

Lemma 15 *For any initial condition, $x_0 = x \in [0, 1] \setminus [x^*, x^{**})$, $\theta_0 = H$ or $x_0 = x$ and $\theta_0 = L$ where x is a point where the firm was investing before and after commitment, in the commitment game $V_c(1, H) \geq V(1, H)$.*

Proof. If x_0 is such that before and after the firm commits the firm is investing at x_0, then the only way the firm's payoffs can increase is if $V_c(1, H) > V(1, H)$. So the initial conditions that matter are initial conditions where the firm is not investing in the commitment game. Suppose that $V(1, H) > V_c(1, H)$, from the previous lemma, it must be that $x^{**} > x_c^{**}$. There are two cases to consider.

If $x_c^* > x^*$[12] then

$$\begin{aligned} c/\lambda &= \int_0^\infty \bar{\pi} e^{-(r+\lambda)t}[V_c(1, H) - V_c(x_c^*, H)]dt \\ &= \int_0^\infty \bar{\pi} e^{-(r+\lambda)t}[\frac{(r+\lambda)}{r+\lambda+\bar{\pi}} V_c(1, H) - \frac{1}{r+\lambda+\bar{\pi}}[x_c^*(1+\lambda/r) - k\bar{\pi}]] \\ &< \int_0^\infty \bar{\pi} e^{-(r+\lambda)t}[\frac{(r+\lambda)}{r+\lambda+\bar{\pi}} V(1, H) - \frac{1}{r+\lambda+\bar{\pi}}[x^*(1+\lambda/r) - k\bar{\pi}]] \\ &= \int_0^\infty \bar{\pi} e^{-(r+\lambda)t}[V(1, H) - V(x^*, H)]dt \\ &= c/\lambda, \end{aligned}$$

which is a contradiction.

If $x^{**} > x_c^{**}$ and $x^* \geq x_c^*$ If $\theta_0 = H$, then $V_c(1, H)$ can be written as

$$V_c(1, H) > \int_0^t e^{-\int_0^s (r+\lambda)d\xi}[x_{c,s} + \lambda V_c(x_s, L)]ds + e^{-\int_0^t (r+\lambda)ds} V_c(x_0, H),$$

[12]if $x^* = 0$, then $V(1, H) > V_c(1, H)$ implies that this condition cannot hold, so I only need to consider the other case.

while So its only possible for $V_c(1,H)$ is lower if $V_c(x_s,L)$ is lower for some x_s.

At x_c^{**}, $V(x_c^{**},\theta) > V_c(x_c^{**},\theta)$ since the investment cutoff is lower and $V_c(1,H)$ is lower. Moreover, $D(x_c^{**}) > D_c(x_c^{**})$, because $V(1,H)$ is larger than $V_c(1,H)$.

$D(x) > D_c(x)$ also holds between x_c^{**} and x^{**}, because one firm is promoting and the other isn't, and the point with this larger difference is reached faster for the firm without commitment, so $V_c(x^{**},L) > V(x^{**},L)$, which must also hold at all points above x^{**}, which is a contradiction. ■

Lemma 16 *If $V_c(1,H) > V(1,H)$, then $x_c^* \geq x^*$*

Proof. Suppose not, then[13]

$$\begin{aligned}
c/\lambda &= \int_0^\infty \bar{\pi} e^{-(r+\lambda)t}[V_c(1,H) - V_c(x_c^*,H)]dt \\
&= \int_0^\infty \bar{\pi} e^{-(r+\lambda)t}[\frac{(r+\lambda)}{r+\lambda+\bar{\pi}}V_c(1,H) - \frac{1}{r+\lambda+\bar{\pi}}(x_c^*(1+\lambda/r) - k\bar{\pi})] \\
&> \int_0^\infty \bar{\pi} e^{-(r+\lambda)t}[\frac{(r+\lambda)}{r+\lambda+\bar{\pi}}V(1,H) - \frac{1}{r+\lambda+\bar{\pi}}(x^*(1+\lambda/r) - k\bar{\pi})] \\
&= \int_0^\infty \bar{\pi} e^{-(r+\lambda)t}[V(1,H) - V(x^*,H)]dt \\
&= c/\lambda,
\end{aligned}$$

This is a contradiction, so $x_c^* \geq x^*$. ■

Properties of Equilibrium

Proof of Lemma 4

As in the good news case, the value functions can be rewritten as

$$V(x,\theta) = \int_0^\infty e^{-\int_0^t (r+\lambda+\bar{\pi}-\pi_s)ds}[x_t - (\bar{\pi}-\pi_t)1_{\theta=L}(\Delta(x_t)) + \lambda a_t D(x_t) + \lambda(V(x_t,L) - V(x_t,\theta)) - k\pi_t - ca_t]ds,$$

which directly gives the sequential rationality conditions. The monotonicity conditions follow from the same logic as in the good news case.

Moreover,
$$D(x) = \int_0^\infty e^{-(r+\lambda)t}(k\pi_t + (\bar{\pi}-\pi_t)\Delta(x_t))dt,$$
which can be seen by taking the difference of the above expression for $V(x,\theta)$.

[13] If $x^* = 0$ (which $x_c^* = 0$ implies), then this holds vacuously, so I am considering the other case.

Existence and Uniqueness

Proof of Proposition 6 and 10:

By the sequential rationality condition, $D(x)$ is bounded above by

$$D(x) \leq \frac{k\bar{\pi}}{r+\lambda},$$

so if $c > \frac{\lambda}{r+\lambda}k\bar{\pi}$, then the firm never invests.

Case 1: $c > \frac{\lambda}{r+\lambda}k\bar{\pi}$.

Now it can never be optimal for the firm to exert effort. It is thus sufficient to find the cutoff where the firm prefers to stop censoring, because for any beliefs the optimal choice of this cutoff will automatically preclude any effort exerted by the firm. Consider the modified problem where x_t follows law of motion $\dot{x}_t = -\lambda x_t + \bar{\pi}x_t(1-x_t)$ if this is negative and $\dot{x}_t = 0$ otherwise and the firm never censors. Denote the value function for this problem as V_{nc}. Then the solution to

$$\max_{x^{**}} \int_0^{x^{**}} (V_{nc}(x, L) - k)dx.$$

exists, and $V_{nc}(x, L) \geq V(x, L)$ for all beliefs below x^{**}, and $V_{nc}(x^{**}, L) = V(x^{**})$. $V_{nc}(\cdot, L)$ is continuous, so $V(x^{**}, L) \geq k$ above x^{**} (or $x^{**} = 1$) and since $V(x, L)$ is increasing in x, this holds for all $x > x^{**}$ so this is sequentially rational. Moreover, since V_{nc} is monotonically increasing, this cutoff is unique. This implies uniqueness of the equilibrium. The uniqueness of proposition 10 follows from exactly the same logic where the modified law of motion is $\dot{x}_t = -\lambda x_t + \mu x_t(1-x_t)$.

Case 2: $c \leq \frac{\lambda}{r+\lambda}k\bar{\pi}$.

There always exists an equilibrium where the firm stops censoring at a higher belief than when it stops working. Consider the following problem.

$$\max_{x^{**}} \int_{x^{**}}^1 \bar{\pi}(V_{wc}(x, L) - k)dx.$$

where V_{wc} is the value function corresponding to the problem where the firm is working and censoring everywhere except at 0 where it is doing nothing and is believed to be doing that. This is well defined, strictly increasing, and continuous except at 0. This maximization problem has a solution, x^{**}.

Define $D_{x^*}(x^*+) = \lim_{\epsilon\to 0} D^{x^*}(x^* + \epsilon)$ and $D_{x^*}(x^*-) = \lim_{\epsilon\to 0} D^{x^*}(x^* - \epsilon)$. As shown in Board and Meyer-ter-Vehn (2013), function $D_{x^*}(x^*\pm)$ satisfies

1. $D_{x^*}(x^*+)$ is continuous and strictly increasing on $[0, x^{**})$ and $\lim_{x^*\to x^{**}} D_{x^*}(x^*+) = \lim_{x^*\to x^{**}} D_0(x^*+)$.

2. $\lim_{x^* \to x^{**}} D_{x^*}(x^*-) = \lim_{x^* \to x^{**}} D_{x^{**}}(x^{**}-)$, and if $\lambda \geq \bar{\pi}$ then $D_{x^*}(x^*-)$ is continuous and strictly increasing with $\lim_{x^* \to 0} D^{x^*}(x^*-) = 0$.

3. At any cutoff x^*, if beliefs are drifting up above x^* and down below x^*, then $D_{x^*}(x^*+) > D_{x^*}(x^*-)$.

Property 1 follows from the following observation. When the firm is believed to be working and not censoring, beliefs are drifting up, and never drop below x^* unless news arrives and they go to 0. So $D_{x^*}(x)$ is the same as $D_0(x)$, the difference between the value of a high and low quality product of a firm who works everywhere but at $x = 0$, for all $x^{**} > x > x^*$. So $D_{x^*}(x^*+) = D_0(x^*+)$, and $D_0(x)$ is strictly increasing and continuous by the previous lemma, and $D_{x^*}(x^{**}+) = D_0(x^{**}+)$.

Property 2 follows from the same logic as property 1. For any cutoff $x^{**} > x^* \geq \max(1 - \lambda/\bar{\pi}, 0)$, beliefs always stay below x^* once they start below x^*, so the firm's payoffs are the same as the payoffs they'd receive if they started shirking at x^{**}, and since $D_{x^{**}}(x)$ is continuous and increasing below x^{**} and above $\max(1 - \lambda/\bar{\pi}, 0)$, so is $D_{x^*}(x^*-)$.

Property 3 follows directly from the monotonicity of $D_{x^*}(x)$. Since this is increasing, it must always be that $D_{x^*}(x^*+) \geq D_{x^*}(x^*-)$ with this holding strictly at a divergent cutoff, where the function must be discontinuous.

A cutoff x^* is consistent with an equilibrium if $D_{x^*}(x^*-) \leq c/\lambda \leq D_{x^*}(x^*+)$ or if $D_0(0+) \geq c/\lambda$. The continuity and monotonicity of $D_{x^*}(x^*+)$ implies that an x^* exists that satisfies the sequential rationality condition, since $D_{x^*}(x^*+)$ must either cross c or always lie above or below c on $[0, x^{**})$, and above x^{**}, for any cutoff $x^* \in [0, x^{**})$ it must be that $D_{x^*}(x) = \frac{1}{r+\lambda} k\bar{\pi} > c/\lambda$, so, the firm always wants to work there.

Finally, since the drift above x^{**} is positive x_t never drops below x^{**} once it has gone above it, so $V(x^{**}, L) = V_{wc}(x^{**}, L)$, where V is the value function when the firm is playing x^{**} and x^* as cutoffs, so x^{**} is still sequentially rational.

In the case of **proposition 10**, the above logic holds. The only exception being that

$$
\begin{aligned}
D(x) &= \int_0^\infty e^{-(r+\lambda)t}(k\pi_t + (\bar{\pi} - \pi_t)\Delta(x_t))dt \\
&\leq \int_0^\infty e^{-(r+\lambda)t} k\bar{\pi} + \epsilon\Delta(x_t) \\
&\leq \frac{k\bar{\pi} + \epsilon}{r + \lambda}.
\end{aligned}
$$

The firm never invests if $c > \frac{\lambda}{r+\lambda}(k\bar{\pi} + \epsilon)$. The existence argument from **Case 1** holds in this case, and the argument from **Case 2** holds when $c < \frac{\lambda}{r+\lambda}(k\bar{\pi}\epsilon)$.

Uniqueness of Perfect Bayesian Equilibrium

Proof of Proposition 8 This proof proceeds in two steps. First, I show that the firm never conditions on their private history, and then argue that this implies that any PBE is the MPE, up to what happens at the point where the firm is indifferent.

Lemma 17 *For any two private histories, h_t and h'_t, it is optimal to play $a(h_t) = a(h'_t)$ if $x(h_t) = x(h'_t)$ and $\pi(h_t) = \pi(h'_t)$ if $x(h_t) = x(h'_t)$ and $\theta(h_t) = \theta(h'_t)$, where $x(\cdot)$ is the current reputation after history h and θ is the current quality after history h, and $a(\cdot)$ and $\pi(\cdot)$ are the continuation strategies (the entire stochastic process from time t forward).*

Proof. Consider any two histories with the same reputation and quality at times t. Suppose that $(a(h_t), \pi(h_t))$ gave a higher payoff than $(a(h'_t), \pi(h'_t))$. Then the firm with history h'_t could deviate to the strategy $(a(h_t), \pi(h_t))$ and receive a higher payoff, since this would induce the same measure over public history, which in turn would induce the same payoffs. So after any two histories that induce the same reputation and type, the firm's strategies can only be different if they are indifferent. ■

Continuation values can thus be written as a function of the public history and the firm's private type at the current instant. A strategy a is sequentially rational at a public history h_t if it maximizes

$$\lambda D(h_t)a_t - ca_t,$$

where $D(h_t) = V(h_t, H) - V(h_t, L)$, and similarly π maximizes

$$\pi(\Delta(h_t) - k).$$

The expression, $D(h_T)$ can be rewritten as

$$D(h_t) = \int e^{-(r+\lambda)}[(\bar{\pi} - \pi_t)\Delta(h_s^{\emptyset}) + k\pi_t]\,dt.$$

where $h_s^{\emptyset}$ is the public history following time t where no bad news has arrived by time s. Since, k is π_t is minimizing the integrand,

$$D(h_t) \leq \frac{1}{r+\lambda}k\bar{\pi},$$

which implies that investment is never sequentially rational. If $\lambda \geq \bar{\pi}$ this directly implies that the MPE is unique, since beliefs must drift down everywhere, so this is markovian.

Otherwise there could be an equilibrium where beliefs oscillate, the firm censors, let beliefs drifts down, and then stops and lets beliefs drift up, or an equilibria where beliefs always drift in the same direction, but the drift is constant for a while and then starts drifting again.

Lemma 18 *Fix a time t. If no news has arrived by time t, and beliefs are monotone in t until news arrives, then continuation values are also monotone in time.*

Proof. Let $V_{t+s,\theta}$ be the continuation value at time $t+s$ when the firm's current quality is θ. From the previous lemma, the current calender time and the current quality pin down continuation payoffs. Suppose x_{t+s} is increasing in s (the argument if it is decreasing is analogous). Then, for any s, s', $s > s'$, the $V_{t+s,\theta} \geq V_{t+s',\theta}$, since beliefs (weakly) higher at every future time $x_{t+s+v} \geq x_{t+s'+v}$ for any $v > 0$, so the firm could mimic the strategy they would play starting at s' and receive exactly the same measure over quality shocks and news. Moreover, the firm's flow payoff would be weakly higher at every point. Therefore the firm's continuation payoffs must be higher. ∎

Lemma 19 *If $\bar{\pi} > \lambda$ then along any trajectory the drift of beliefs can never change directions.*

Proof. The case where beliefs drift for a while, stop, and then start drifting again in the same direction is impossible, because at the constant point, the agent would have to be indifferent between censoring and not and the value functions would be monotone because the drift of beliefs does not change sign, which in turn implies that it would never be sequentially rational to take an action that makes beliefs start drifting again after they are constant. What remains is to rule out is cases where the drift changes direction.

Suppose that belief's oscillate, or the drift switches directions. First, suppose that the drift just switches directions once, from down to up (the argument when they switch from up to down is analogous). Then let t be a time such that beliefs are drifting up at time t and there exists a $t' < t$ such that $x_t = x_{t'}$ and beliefs are drifting down at t'. Then, the firm must be censoring at t' and not censoring at t. But, since the drift never switches sign after time t, it must be that $x_{t+s} > x_{t'+s}$ for any $s > 0$, which implies that $k \geq \Delta_t > \Delta_{t'} \geq k$, which is a contradiction.

It remains to rule out oscillations. Suppose beliefs drift down, then up, then down again (the argument for up, down, up is analogous). Then we can find a time t where beliefs are drifting down at x_t, $t < t'$ where beliefs are drifting up at $x_{t'}$, $x_t = x_{t'}$ and the drift only switches signs once between t and t'. Let s be the time such that $t' + s$ is the time when the drift switches signs again. We can always find t and t' such that beliefs are drifting up at $t + s$. Then for any $0 < u < s$, $x_{t+u} < x_{t'+u}$, and the firm must not be censoring at time $t + s$. So $V_{t+s}(L) < V_{t'+s}(L)$. Since this is true and $x_{t+u} > x_{t'+u}$ at every point, the firm's payoff must be strictly higher at t', but the firm is censoring at t and not at t' so $k \geq \Delta_{t'} > \Delta_t \geq k$. ∎

Therefore, in all equilibria, beliefs are either drifting in one direction or are constant until an arrival happens. The only way for beliefs to be constant is if the firm is indifferent between censoring and not. This implies there exists Markovian strategies that induce this law of motion for beliefs, and only differ from the perfect Bayesian strategies at indifference points, so these strategies also induce the same payoffs. Since the Markov perfect equilibrium is unique, these must be the Markov strategies from proposition 7.

Commitment

Proof of Corollary 3 Consider a sequence $r_n \to 0$. In this sequence of equilibrium

$$\lim_{n\to\infty} r_n(V_n(x,\theta) + E(\int e^{-r_n t} k\pi_t \, dt)) = \lim_{n\to\infty} E\left(r_n \int_0^\infty e^{-r_n t} x_t \, dt\right) = 0,$$

since

$$\begin{aligned} r_n \int_0^\infty e^{r_n t} x_t \, dt &= r_n[\int_0^T e^{-r_n t} x_t \, dt + \int_T^\infty e^{-(r_n+\mu)t} x_t \, dt], \\ &= r_n[\int_0^T x e^{-(r_n+\lambda)t \, dt} + \int_T^\infty e^{-(r_n+\mu)t} x_t \, dt] \\ &= 0 \end{aligned}$$

where T is the time it takes to reach the region where the firm doesn't censor.

On the other hand, if the firm has committed to not censor, for any x where $x > \min(\frac{\lambda}{\pi} - 1, \limsup x^*_{c,n})$, there exists an N such that for any $n \geq N$, beliefs are drifting up in all equilibria in the sequence at that point. Finally

$$r_n V_n(x, H) = r_n \int e^{-r_n t}[x_t - c]dt > 0.$$

and

$$r_n V_n(x, L) = r_n \int e^{-(r_n+\lambda)t}[x_t - c + \lambda V_n(x, H)]dt \geq \frac{\lambda}{r_n + \lambda}(r_n V_n(x^*_{c,n}, H)) > 0,$$

so the firm always benefits from commitment.

Exogenous Good News

Proof of Proposition 9

The argument from lemma 2 and lemma 1 are identical in this environment. So $D(x)$ is decreasing and $\Delta(x)$ is decreasing. This implies that any MPE can be characterized by two cutoffs. If $\lambda \geq \mu$, then x^* is an absorbing cutoff. Once beliefs reach x^*, they stay there until news arrives. This implies that, in any MPE

$$D(x^*) = \int e^{-(r+\lambda)t}[(\mu)\Delta(x^*) + \pi_t(\Delta(x^*) - k)]dt.$$

Case 1: $\left(1 + \frac{r}{\lambda}\right) c \geq k\mu$ and $\lambda \geq \mu$. Suppose that $x^* > x^{**}$. Then:

$$\frac{c}{\lambda} \geq D(x^*) = \frac{1}{r+\lambda}\mu\Delta(x^*) < \frac{k\mu}{r+\lambda},$$

since the firm does not find promotion to be optimal at this point. But, this contradicts the assumption that $\left(1+\frac{r}{\lambda}\right)c \geq k\mu$, so this cannot be an equilibrium.

Case 2: $\left(1+\frac{r}{\lambda}\right)c < k\mu$ and $\lambda \geq \mu$. Suppose that $x^{**} \geq x^* > 0$. Then

$$c/\lambda \geq D(x^*) \geq D(x^{**}) \geq \int e^{-(r+\lambda)}\mu\Delta(x^{**})\, dt \geq \frac{\mu k}{r+\lambda}.$$

if $x^{**} > 0$, which is a contradiction. If $x^* = 0$, then by similar logic, $k \geq \Delta(0)$, so $x^{**} = 0$.