# A TALMUDIC BANKRUPTCY SOLUTION: THE CCC PRINCIPLE

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**VINCY FON**[1]
**THE GEORGE WASHINGTON UNIVERSITY**

*Abstract*. Following a bankruptcy, how should we distribute the available assets among the eligible creditors? Most people would accept a proportional distribution—for each claimant, calculate her percentage of the sum of all claims and assign her that same percentage of total assets. However, this is not the only reasonable approach. For example, if every claim is at least as large as total assets, assigning an equal share to every creditor is a sensible solution. A set of three numerical bankruptcy examples for three claimants, discussed 2,000 years ago in the Talmud, coincide with the above two approaches, but the third case remained a puzzle until recently when modern game theory (Aumann and Maschler 1985) was enlisted to demystify all cases. This paper explains the unifying principle, the *Contested-Claim Consistency* principle (CCC), behind the Talmudic examples. Importantly, it uses different means to better understand the logic behind the CCC bankruptcy allocations and points out the subtle yet important properties behind them. This clarifies the meaning of fairness underlying the CCC allocation, and may better convey the meaning of the *Pari Passu* provision that appears in many existing International Sovereign Debt Instruments.

*Keywords*: Bankruptcy, CCC principle, Contested Garment, Contested-Garment Consistent, CG-consistent, Contested-Claim Consistent, Nucleolus, *pari passu*

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[1] The George Washington University, Department of Economics. Email: vfon@gwu.edu. Special thanks are due Dan Milkove for his comments and help. All remaining errors are mine.

## 1. Introduction

It took 2,000 years and two illustrious modern game theorists to crack the following bankruptcy solution from the Babylonian Talmud. Three creditors with claims of 100, 200, and 300 are to split the bankrupt estate. When the estate value is 100, the creditors split the estate evenly at 33 and 1/3 apiece. For an estate worth 200, the creditors get 50, 75, and 75. When the estate value is 300, the creditors receive 50, 100, and 150, in proportion to their claims. These lessons, attributed to Rabbi Nathan, befuddled scholars over the millennia.[2] When the estate is small or large, the even split and the proportional split seem reasonable, considering each case independently. But no one could find a reasonable explanation of the bankruptcy division when the estate is worth 200. Putting all three cases together is even more mysterious; the underlying principles dictating the numerical examples were thus hidden in an abyss for ages.

In the mid-nineteen-eighties, Robert Aumann and Michael Maschler unexpectedly discovered that the Nathan examples prescribe the same solutions as those from the nucleoli of the corresponding coalitional games.[3] Not armed with modern game theoretical concepts, it is inconceivable that the wise sage could develop the bankruptcy solution through the same means. The two distinguished scholars were then inspired to find alternative mechanisms by which Rabbi Nathan came to his numerical lessons. Aided by research on nucleolus, Aumann and Maschler discovered that the Nathan solutions can be explained through the consistent application of another Talmudic principle – the Contested Garment principle.[4] The game theorists stated that this discovery was possible only after they uncovered the relation between the nucleoli and the Nathan examples. In their 1985 *Journal of Economic Theory* article, Aumann and Maschler present a precise definition for a non-game-theoretic bankruptcy solution, which generalizes the numerical Talmudic examples. They call it the Contested-Garment Consistent principle, or the CG-consistent principle.[5] [6] The term “garment” in the name of the principle refers to *claims* in the bankruptcy problem. To respect the historical source while giving it a modern flavor, we interchange contested garment with contested claim in this article.

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[2] The Nathan examples are presented in the Tractate of *Kethuboth* 93a (chap.10, mishnah 4).

[3] In general, a coalitional game centers its attention on coalitions formed by subsets of players in a game. The nucleolus further focuses on dissatisfaction for any coalition; it is the solution that minimizes the largest dissatisfaction among all possible coalitions (Hill 2000). The nucleolus of a coalitional game exists and is unique, and it is group- as well as individually-rational.

[4] The Contested Garment example is presented in the Tractate of *Bava Metzia* (2a) of the Talmud.

[5] Aumann and Maschler (1985) actually presented “three different justifications of the solution to the bankruptcy problem that the nucleolus prescribes…” However, most attention is paid to the bankruptcy solution derived by applying the CG-consistent principle.

[6] This Contested-Garment consistent principle is based on the Talmudic example showing how a piece of contested garment should be divided between two creditors. Hence the term “garment.” Specifically, the Contested-Garment consistent principle requires the consistent application of the contested-garment principle for *any pair of creditors*.

For ease of discussion, the Contested-Garment Consistent principle will be renamed the CCC principle, an abbreviation for the *Contested-Claim Consistent* principle.

Aumann and Maschler proved that the CCC principle leads to a unique solution for all bankruptcy problems; the unique division matches each of the solutions in Nathan's three numerical examples, and also coincides with the nucleolus of the properly defined coalitional games. The mathematical, non-game-theoretical presentation of the CCC principle appears straightforward, but the general solution to any bankruptcy scenario is not easy to find and the subtle properties of the bankruptcy divisions are often elusive. Not surprisingly, the little understood CCC principle was not widely applied or studied. Quite apart from the importance of understanding this age-old bankruptcy solution, it is hoped that a better understanding will make the CCC principle a strong alternative to the ubiquitous proportional principle in some bankruptcy cases. Further, it will be exciting to see how a deeper understanding of an age old allocation problem can shed light on recent emerging sovereign bankruptcy problems. For over 100 years, the international debt instrument had incorporated the *Pari Passu* clause, even though its exact meaning was not clear.[7] Recent court cases involving Peru and Argentina accepted the interpretation of *Pari Passu* as a proportional allocation.[8] Elsewhere, we will argue that the CCC principle may better fit the underlying idea of *Pari Passu*.[9]

This article pushes the envelope in many different directions to supplement the pioneering work and the path-breaking insights of Aumann and Maschler, to further our understanding of the Talmudic bankruptcy solution, and to search for the basis upon which the current international sovereign debt crisis may be reasonably connected to the CCC principle. Section 2 reiterates the important discovery by Aumann and Maschler of the CCC principle, employing the non-game theoretic approach. Section 3 presents an alternative definition, one that was briefly mentioned but not well-developed in the Aumann and Maschler article. The equivalence of the two approaches is shown. Section 4 discusses the difference between (1) nominal gains and nominal losses across creditors, to which the literature pays attention, and (2) gains and losses under appropriate bounds, referred to as *de facto* gains and *de facto* losses, as stressed in this article. The alternative way of viewing gains and losses is in fact more revealing; it provides a new understanding of the criterion behind the CCC divisions.

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[7] The Latin phrase *pari passu* means "in equal step."

[8] See: *Elliott Assocs., L.P.*, General Docket No. 2000/QR/92 (Court of Appeals of Brussels, 8th Chamber, 26th Sept. 2000) and the many court cases concerning Argentina such as *NML Capital v. Republic of Argentina*, 621 F.3d 230 (2d Cir. 2010).

[9] Investigating whether the principle fits the meaning of *Pari Passu* better than the alternative proportional allocation inspired the study of the CCC principle in this paper. Two other papers and the current one form a trio to investigate this question. Fon (2015*a*) compares the CCC and the proportional principles. Fon (2015*b*) argues in detail why the CCC principle should be embedded in the *Pari Passu* clause in the international sovereign debt instruments.

The remainder of the article concerns the pattern of CCC bankruptcy allocations. Although an implementation of finding CCC allocations is known and somewhat easy to understand, especially for cases with few creditors, a concise table of divisions for any amounts of claim and estate values has not appeared in the literature.[10] Section 5 establishes a table showing CCC shares assigned to each creditor for the building-block case of two creditors with any claim and estate values, highlighting the boundary scenarios. Section 6 proves and extends the CCC division for the case of three creditors. The table is then applied to the historical case discussed by Rabbi Nathan, in which the claims owed to the creditors are 100, 200, and 300. The table shows the assignment of the CCC divisions for *any* value of the estate; providing a complete answer to the millennia-old puzzle and extending other numerical tables in the literature.

Section 7 concludes the investigation by explaining the different aspects of the CCC bankruptcy allocations, and then briefly explores the potential application of the CCC divisions to the current *Pari Passu* clause commonly employed in international sovereign debt instruments. The interest in the meaning of *Pari Passu* began in the year 2000 with a case concerning Peruvian debt issues in the Belgium court, and continues to play an important role in the recent Argentinian debt default cases being battled in New York and British courts. Considering that many outstanding sovereign instruments with the *Pari Passu* provision are around and that many countries are at risk financially, it is time to investigate exactly what *Pari Passu* should mean and whether it makes sense to apply the CCC principle to the provision. The paper will end with some thoughts on this issue. But first, we must fully understand what is behind the simple yet mysterious idea of CCC.

## 2. The CCC Principle and the Equal Division of Contested Claim

The appropriate way to start our investigation is to reiterate the Talmudic bankruptcy lesson attributed to Rabbi Nathan. The numerical recommendations are presented in Table 1, where rows present each claimant's share of the bankruptcy allocation, and the columns present different values for the bankrupt estate. The common lesson embedded in these three numerical examples is murky: it is not clear how the allocation scheme evolves from an even-split with a small estate value to a proportional division when the estate gradually increases in value. Exactly what rule dictates the division for an estate worth 200? Further, what happens when the estate increases beyond 300?

[10] Kaminski (2000) proposes a physical solution to the problem using a hydraulic analogy; this approach can be visualized with the help of graphs.

| Claim \ Estate | 100 | 200 | 300 |
|---|---|---|---|
| **100** | 33 1/3 | 50 | 50 |
| **200** | 33 1/3 | 75 | 100 |
| **300** | 33 1/3 | 75 | 150 |

***Table 1. Rabbi Nathan's Recommendation on the Bankruptcy Problem***

The mysterious lessons deterred the application of this principle for two millennia. In the 1980s, Aumann and Maschler (A&M) translated the three bankruptcy examples into game-theoretic models, tested them against all known solutions, and discovered that only the nucleolus, a solution concept invented by the game theorist David Schmeidler (1969), generates exactly the same divisions as the Nathan examples. A&M (1985) present three different non-game theoretic explanations to identify the bankruptcy solution. We follow the subsequent literature by centering our attention on their first and most approachable characterization.[11] A&M found that two principles are behind the hidden lessons. The first principle allocates a deficit sum between two creditors, and then the basic allocation principle is consistently applied to any pair of creditors. While the consistency requirement is nowhere near transparent from Nathan's numerical examples, the 2-creditor building block case advocated by A&M resonates another easily understood Talmudic lesson, the contested garment (the contested claim) principle. According to this principle, "Two hold a garment; one claims it all, the other claims half. Then the one is awarded three-fourths, the other one-fourth."[12] Since the lesser claimant only claims half of the garment/estate, she concedes half to the greater claimant.[13] As the contested claim only involves half of the estate, the two claimants equally share this, leaving the greater claimant with three-quarters (one half plus one quarter) and the lesser claimant with one-quarter.[14] Thus, the "contested claim principle" always implies equal sharing between the creditors.

This contested claim principle is easy to understand, but the reason why Nathan's unifying lesson eluded investigative attacks throughout the Ages lies in the *consistent* application of the contested claim principle for any pair of creditors. We combine the two requirements and call it the Contested-Claim Consistent (CCC) principle. Intuitively, consistency means that any two creditors always divide the total amount assigned to them by applying the contested claim

[11] For example, see Aumann (2002); Elishakoff (2011), and Schecter (2012).

[12] The Tractate of Bava Metzia (2a) in the Talmud starts off with this contested garment example.

[13] We use "she" to refer to creditors. This is done in deference to the Talmudic case law concerning bankruptcy which inspires the CCC principle; creditors were women in the classic examples.

[14] Aumann (2002) attributes this to Rashi, who "explains here that the claimant to half the garment 'concedes…that half belongs to the other, so that the dispute revolves solely around the other half. Consequently, … each of them receives half of the disputed amount.' "

principle. Specifically, taking the sum of the amounts given to two creditors as "the value of an estate," the contested claim principle is applied to divide this sum between these two creditors. Or, the division of a bankrupt estate among any number of creditors is such that any two of them always divide the sum they jointly receive according to the principle of equal sharing of contested claim.

Although it is extremely difficult to detect from the Talmudic numerical examples, the consistency requirement is easy to grasp and confirm once the bankruptcy division is proposed. To see clearly that the CCC principle is applied throughout Nathan's examples, first take the allocative case when the estate is 100. Any two creditors are jointly awarded 66.66. Since either creditor's claim exceeds this amount, nothing is conceded and all of 66.66 are contested. Sharing this contested claim equally between them, both creditors receive 33.33, as stated. Next, for an estate worth 200, take the simple case between the 200- and the 300-claimants. Jointly they receive the sum of 150, which falls below both claims. So the entire 150 is contested and split equally, leading to 75 to be awarded to each claimant. Now consider what happens between the 100-claimant and either of the remaining claimants. In each case the two claimants are assigned 125, where the sum exceeds the claim of the lesser claimant but not the greater claimant. The 100-claimant concedes 25 to the other claimant and contests 100 of the estate, while the other claimant concedes nothing to the 100-claimant and contests the whole estate. This means that the contested claim of 100 is equally divided between the two claimants, giving 50 to the 100-claimant and 25+50 to the greater claimant. Lastly, for an estate value of 300, it is straightforward to confirm that the shares awarded to each pair of claimants follow the CCC principle, where the jointly awarded sum for any pair of claimants always falls between the two claims.

As noted, when a CCC allocation is known, confirming that the CCC principle is consistently applied is straightforward. On the other hand, even after it is pointed out that the CCC principle should be applied, finding the CCC allocation can often be daunting, especially with many creditors. We will return to this practical issue later in the article. Fundamentally, once the mystery of consistency is resolved, understanding the underlying characteristics of the CCC allocations lies solely in the basic case of equal division of contested claim (CC) by any 2 creditors. Thus, to fully comprehend the CCC principle, we need to carefully study the building-block case in which two creditors share a bankrupt estate.

Mathematically, A&M describe a 2-creditor problem with estate $E$ and claims $d_1$ and $d_2$ (with $d_1 \leq d_2$) by first defining the concession offered to each claimant, and then adding the conceded amount with half of the residual of the two concessions as the share ($s_i$) awarded to each creditor. Formally, let $c_i^{\sim} = \max\{E - d_j, 0\}$ be the amount conceded by creditor $j$ in favor of creditor $i$.[15] We

[15] Note the two subscripts embedded in the definition of the concession to creditor $i$ from the other creditor $j$.

will refer to $c_i^{\sim}$ as the *de facto* concession awarded to $i$ (by the other claimant $j$). Intuitively, a positive amount is conceded to $i$ only if creditor $j$'s claim is less than the estate value $E$; the *de facto* concession is zero if $j$'s claim is greater than the estate value because concessions cannot be negative. More specifically, when $E>d_j$, the *de fa*cto concession $c_i^{\sim} = E–d_j$ is positive and it matches our intuition; when $E<d_j$, the *de fa*cto concession $c_i^{\sim}$ is nil. Given the *de facto* concessions, the disputed amount to be shared equally is $E–c_1^{\sim}–c_2^{\sim}$, and the share given to creditor $i$ under the equal division of the contested claim principle is $s_i = c_i^{\sim}+(E–c_1^{\sim}–c_2^{\sim})/2$.[16]

When A&M abstract from the Talmudic lesson of equal sharing of contested claim and transpose the underlying principle to mathematical expressions, they incorporate two subtle extensions. The first is that in applying the Talmudic principle, any amount claimed beyond the estate value by a creditor is irrelevant; only the fact that the claim is at least as large as the estate is important.[17] This is incorporated in the definition of the *de facto* concession $c_i^{\sim} = \max\{E–d_j,0\}$, in which whenever a claim $d_j$ is greater than the estate $E$, the negative difference $E–d_j$ must be replaced by 0. Equivalently, any claim exceeding the worth of the estate, no matter how large, is treated as if the claim equals the size of the estate, because it is pointless to dispute any amount of claim beyond the value of the available estate. This makes sense even though the historic Talmudic garment example does not showcase the case in which a claim exceeds the entire garment/estate; our focus is on the sharing aspect between the creditors.

The second subtle extension established by A&M lies in the concept of contested claim and the naming of it. In the original contested garment example, one creditor claims the whole garment while the other claims half, making the idea of taking the contested claim to be half of the garment pretty easy to accept. However, a little reflection suggests that the concept of contested claim may not be unique.[18] This is because whenever each individual claim is less than the estate, both claims can be considered contested. For example, if the creditors' claims are 1/2 and 2/3 of the estate, one can argue that both 1/2 and 2/3 are good candidates for contested claim (by one creditor only).[19] When a unique and appropriate concept for contested claim by both

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[16] For ease of extending the A&M (1985) mathematical definition for the contested-claim allocations, we introduce a slightly different notation $c_i^{\sim}$ to denote concessions and highlight the restriction that concessions cannot be negative by calling them *de facto* concessions. A&M call the amount allocated to a creditor an "award;" we mostly call the same concept a "share." This is done because award gives a positive connotation, while the forthcoming alternative definition provided in this article concentrates on loss suffered, which conveys a negative connotation. Further, note the usage of the term share; it represents the amount but not the proportion of the bankrupt estate given to a creditor.

[17] A&M (1985) state this fact clearly at the beginning of their article: "Any amount of debt to one person that goes beyond the entire estate might well be considered irrelevant; you cannot get more than there is."

[18] In each of Nathan's examples, to divide what a pair of creditors receive together in accordance with the CCC principle, either no concession or one concession is involved. In these cases, the concept of contested claim is straightforward.

[19] One might surmise that a good candidate for contested claim would be $\min\{d_1,d_2\}$, which seems to be compatible with the Talmudic garment example. Let us take contested claim to equal $\min\{d_1,d_2\}$. Consider what happens when the estate is larger than both claims: $d_1<d_2<E$. Concessions then equal $E– d_2$ and $E– d_1$, and half of the contested

creditors is lacking, it means that the crucial concept involved is actually not a contested sum. In fact, as A&M point out, the crucial concept is the residual of the two (*de facto*) concessions from the estate. This residual is what should be divided equally and be added to the *de facto* concession offered by the opponent creditor. Thus, it is important to recognize that the term "contested claim" is but a short-hand for the residual of *de facto* concessions from the estate.

A&M did not take the residual-definition rabbit out of a hat. Their precise definition has the backing of another Talmudic source,[20] which was cited in Aumann (2002). Without getting into too much detail, a grandfather dies and is survived by three grandsons. One troublesome grandson claims 1/2 of his grandfather's estate while a coalition of two other grandsons jointly claims 2/3 of the grandfather's inheritance. Thus, 1/2 of the estate is conceded by the troublesome grandson to the coalition of the other grandsons, and 1/3 of the estate is conceded by the coalition to the troublesome grandson. This leaves 1/6 of the estate to be split equally between the troublesome grandson and the coalition, and the troublesome grandson ends up with 5/12 (equals 1/3+1/12) while the coalition of grandsons receives 7/12 (equals 1/2+1/12) of the estate. The theoretical definition given by A&M describes exactly this division of the residual of the concessions from the estate.

The two small subtle extensions made by A&M (1985) are reasonable and the extensions retain the original important spirit of equal division of the part of the estate that both creditors considered hers. The shares given to the two creditors under the principle of equal sharing of residual of *de facto* concessions, according to Aumann and Maschler, are:
$s_1^G = c_1^{\sim}+[E–c_2^{\sim}–c_1^{\sim}]/2$, and
$s_2^G = c_2^{\sim}+[E–c_2^{\sim}–c_1^{\sim}]/2$.

The superscript $G$, as in Gain, is inserted here to highlight the gain-sharing approach of the CCC allocations advocated by A&M. The explicit solutions of the creditors' shares, as functions of the estate $E$ and the creditors' claims $d_1$ and $d_2$, will be presented in the next section, after providing an alternative definition of the CCC allocation.

## 3. The CCC Principle and the Equal Division of Joint Loss

We proceed by providing an alternative definition of the CCC division discussed in the last section. Previously, for any pair of creditors the CCC allocation specifies an equal division of

claim is designated as $d_1/2$. The shares allocated to the two creditors then total $[E– d_2+ d_1/2]+[E– d_1+ d_1/2] = 2E– d_2$. But in general this sum does not equal the estate $E$, which should always be the case. We conclude that contested claim cannot be defined as the minimum of the two claims.

[20] This appears in the Tractate of *Bava Metzia* in the Tosefta (a secondary source contemporaneous with the Mishna).

the possible recovery from the bankrupt estate: the residual of the concessions from the estate. What if we flip our attention? Instead of studying how the pair of creditors equally share the excess of the bankrupt estate beyond the conceded amounts, what if the creditors equally share the loss, the shortage of the estate from the total claim?[21] Will the same allocation be obtained? As before, to focus on loss, it is important to recognize that the relevant concepts here are not the shortages of the nominal (the value of) claims $d_i$ from the estate, but the shortages of the *de facto* claims with appropriate bounds. Much akin to the need to bound each creditor's *de facto* concession from below, we need to bound each creditor's claim from above. Previously, an individual *de facto* concession to creditor $i$ is defined as $c_i^{\sim} = \max\{E–d_j,0\}$; presently, an individual *de facto* claim from creditor $i$ is defined as $d_i^{\sim} = \min\{d_i, E\}$.[22] Specifically, when $d_i$ exceeds $E$, it is pointless for creditor $i$ to claim anything more than $E$, so the *de facto* claim for creditor $i$ is capped at the value of the estate: $d_i^{\sim} = E$. When $i$'s claim $d_i$ is less than the estate value $E$, the creditor cannot ask for anything beyond what the estate owes her, so the nominal claim becomes the *de facto* claim: $d_i^{\sim} = d_i$.

Given the *de facto* claims of creditors 1 and 2, the *de facto* joint loss under bankruptcy to be borne by the pair is now $d_1^{\sim}+d_2^{\sim}–E$. Similar to their prominent derivation of a CCC allocation, where a creditor's individual gain is added to the *de facto* concession, A&M also implied that the CCC allocation can be obtained by equally dividing the *de facto* joint loss from the creditors' claims.[23] Here, the individual loss should be subtracted from her *de facto* claim, not from her nominal claim. Again, the *de facto* concept for the claims must be incorporated because claiming anything that is not there has no practical consequence. Thus, under the principle of equal sharing of *de facto* joint loss, the shares (with a superscript $L$ as in Loss) of the bankrupt estate allotted to the creditors, presented as the *de facto* net loss of each creditor, are given below:

$s_1^L = d_1^{\sim}–[d_1^{\sim}–d_2^{\sim}– E]/2$ and
$s_2^L = d_2^{\sim}–[d_1^{\sim}–d_2^{\sim}– E]/2$.

[21] In the beginning of their article, when they explain the principle behind the classic contested garment example in which one claims all and the other claims half, A&M (1985, Footnote 6) point out that "Alternatively, … the loss is shared equally." As we shall see, the idea that loss, without any qualifications, is equally shared is correct only if each creditor's claim is no greater than the estate. This requirement holds in this contested garment example.

[22] As noted earlier, the fact that whenever her claim exceeds the estate, a creditor should adjust her claim to the total estate available was incorporated in the A&M definition of the *de facto* concession. Although no discussion is offered, the concept of *de facto* claim is mentioned in Footnote 8 of A&M (1985).

[23] A&M (1985, Footnote 8) state, "Alternatively, one may argue that neither claimant $i$ can ask for more than $\min(E,d_i)$. If each claimant is awarded this amount, the total payment *may* exceed the estate; the excess is deducted in equal shares from the claimants' *awards*. This procedure leads to the same payoff …". [Italics are added by this author.] This Footnote is correct in spirit but is slightly confusing. We suggest that the second sentence in the quote should be changed to: "If each claimant is awarded this amount, the total payment <u>would</u> exceed the estate <u>under bankruptcy</u>; the excess is deducted in equal shares from the claimants' <u>*de facto* debt</u>, <u>$\min(E,d_i)$</u>." Here, the underlined words replace the words in italics in the original Footnote.

Note the symmetries between the definitions for the shares given to each creditor under the equal sharing of the *de facto* contested claim $s_i^G$ in the last section, and under the equal sharing of *de facto* total loss $s_i^L$ in this section. Since the relative magnitude of an individual claim against the entire estate heavily influences the individual *de facto* concession and *de facto* claim, the shares assigned to the creditors ought to depend crucially on the relations among the estate and the claims as well. Thus, to confirm that the alternative definition assigns the same allocations to the creditors as the A&M definition, Tables 2 and 3 in the following are classified according to the different ranges of $E$.

| | $c_1^{\sim}$ | $c_2^{\sim}$ | $E–c_1^{\sim}–c_2^{\sim}$ | $s_1^G = c_1^{\sim}+(E–c_1^{\sim}–c_2^{\sim})/2$ | $s_2^G = c_2^{\sim}+(E–c_1^{\sim}–c_2^{\sim})/2$ |
|---|---|---|---|---|---|
| $E \le d_1 \le d_2$ | 0 | 0 | $E$ | $E/2$ | $E/2$ |
| $d_1 \le E \le d_2$ | 0 | $E–d_1$ | $d_1$ | $d_1/2$ | $E–d_1/2$ |
| $d_1 \le d_2 \le E$ | $E–d_2$ | $E–d_1$ | $d_1+d_2–E$ | $(E+d_1–d_2)/2$ | $(E+d_2–d_1)/2$ |

Note: $c_1^{\sim} = \max\{E–d_2,0\}$; $c_2^{\sim} = \max\{E–d_1,0\}$.

***Table 2. 2-Creditor Bankruptcy Allocation ($s_1^G$, $s_2^G$) under Equal Sharing of Contested Claim (Aumann and Maschler's definition)***

| | $d_1^{\sim}$ | $d_2^{\sim}$ | $d_1^{\sim}+d_2^{\sim}–E$ | $s_1^L = d_1^{\sim}–(d_1^{\sim}+d_2^{\sim}–E)/2$ | $s_2^L = d_2^{\sim}–(d_1^{\sim}+d_2^{\sim}–E)/2$ |
|---|---|---|---|---|---|
| $E \le d_1 \le d_2$ | $E$ | $E$ | $E$ | $E/2$ | $E/2$ |
| $d_1 \le E \le d_2$ | $d_1$ | $E$ | $d_1$ | $d_1/2$ | $E–d_1/2$ |
| $d_1 \le d_2 \le E$ | $d_1$ | $d_2$ | $d_1+d_2–E$ | $(E+d_1–d_2)/2$ | $(E+d_2–d_1)/2$ |

Note: $d_1^{\sim} = \min\{d_1, E\}$; $d_2^{\sim} = \min\{d_2, E\}$.

***Table 3. 2-Creditor Bankruptcy Allocation ($s_1^L$, $s_2^L$) under Equal Sharing of De Facto Total Loss (Alternative definition)***

Comparing the corresponding entries in the last two columns of Tables 2 and 3 confirms that for all bankrupt estates $E$ with individual claims $d_1$ and $d_2$, we have:

$$s_i^G = c_i^{\sim} + (E–c_1^{\sim}–c_2^{\sim})/2 = s_i^L = d_i^{\sim} – (d_1^{\sim}+d_2^{\sim}–E)/2.$$

Thus, the proposed requirement of equal sharing of *de facto* total loss and the A&M requirement of equal sharing of *de facto* contested claim lead to the same allocations assigned to the two creditors. While this equivalence was alluded to previously and may not be surprising, we will continue to analyze these approaches through a different angle to make clear that the two alternatives are but different views of the same coin, one inspecting it from the top and the other from the bottom. Before that, note that taking the creditors' claims as fixed, each creditor's share is an increasing (but not strictly increasing) function of $E$. When the available funds of the bankrupt estate increases, each creditor should expect to recover more, and certainly not less, of her money. Also, it is intuitive and embedded in the solutions that a creditor with a larger claim should receive no less a repayment than a creditor with a smaller claim.

Besides the identical bankruptcy allocations resulting from the alternative requirements, two important relations embedded in Tables 2 and 3 can be confirmed. The first and more important one is that the *de facto* concession to claimant *i* (from *j*) and the *de facto* claim for creditor *j* are *E*-complements. To observe that, concentrate on a specific range of the estate *E* (say the second row). Note the sum of the variable in column 1 of one Table and the variable in column 2 of the other Table always equals *E*. More specifically, $c_1^{\sim}+d_2^{\sim}=E$ and $c_2^{\sim}+d_1^{\sim}=E$; we shall refer to these as the *E*-complementarity conditions. While these relations may be slightly harder to detect across the two Tables, they are almost definitional. Consider first the case in which the claim of creditor *j* is less than the estate value ($d_j<E$). The *de facto* concession to *i* from *j* is then positive and $c_i^{\sim}=\max\{E–d_j,0\}=E–d_j$, while the *de facto* claim of *j* is the nominal claim, $d_j^{\sim}=\min\{d_j, E\}=d_j$. This renders $c_i^{\sim}+d_j^{\sim}=E$. In the case in which the claim of creditor *j* is greater than the estate value ($d_j>E$), the *de facto* concession to *i* from *j* is zero, $c_i^{\sim}=\max\{E–d_j,0\}=0$, and the *de facto* claim of *j* is reduced to the value of the estate, $d_j^{\sim}=\min\{d_j, E\}=E$. Again, these values confirm the complementary relation of $c_i^{\sim}$ and $d_j^{\sim}$ within the range of *E*: $c_1^{\sim}=E–d_2^{\sim}$ and $c_2^{\sim}=E–d_1^{\sim}$. With *de facto* refinements, the concession due one creditor is the residual of the estate over the other's claim. Roughly, the complementary relation is much like *j* having a piece of *E*-pie. If *j* is not very hungry and the pie is too big, she would offer what she cannot eat to *i*. If *j* is very hungry and the *E*-pie is too small to satisfy her appetite, she can only consume the whole pie herself and would offer nothing to *i* (but she cannot unilaterally force *i* to give her some of *i*'s pie).

Another important relation easily observed from Tables 2 and 3 is that the total gain beyond the *de facto* concessions from the bankrupt estate, $E–c_1^{\sim}–c_2^{\sim}$, and the total *de facto* loss due to bankruptcy, $d_1^{\sim}+d_2^{\sim}–E$, are always equal. That is, the recoverable gain to be equally shared by creditors, as specified in A&M (1985), can be thought of as an irretrievable loss to be borne equally by both. The alternative interpretations can be deduced easily from the pair of *E*-complementarity relations: summing $c_1^{\sim}+d_2^{\sim}=E$ and $c_2^{\sim}+d_1^{\sim}=E$ gives $c_1^{\sim}+c_2^{\sim}+d_1^{\sim}+d_2^{\sim}=2E$; rearranging this equation gives $E–c_1^{\sim}–c_2^{\sim}=d_1^{\sim}+d_2^{\sim}–E$. It is notable that the recoverable *de facto* gain and its alternative interpretation, the irretrievable *de facto* loss, are always positive. (They take the value of either the estate or the smaller claim if the bankrupt estate is less than at least one creditor's claim, and they equal $d_1+d_2–E$ when the estate is greater than both claims, in which case it is positive because of bankruptcy.)

It is noteworthy that the *E*-complementarity conditions imply that the total *de facto* loss $d_1^{\sim}+d_2^{\sim}–E$ can be interpreted as $d_1^{\sim}–c_1^{\sim}$ or $d_2^{\sim}–c_2^{\sim}$. These two results illustrate precisely something that is intuitive: the maximum potential loss for each creditor is the same if the creditor has to bear the burden alone. It is given by the (*de facto*) total loss facing both creditors together, $d_1^{\sim}+d_2^{\sim}–E$. Since $E–c_1^{\sim}–c_2^{\sim}=d_1^{\sim}+d_2^{\sim}–E$ is always positive under bankruptcy, we now have the following:

$$E–c_1^{\sim}–c_2^{\sim} = d_1^{\sim}+d_2^{\sim}–E = d_1^{\sim}–c_1^{\sim} = d_2^{\sim}–c_2^{\sim} > 0.$$

These relations indicate that the distance from $c_i^{\sim}$ to $d_i^{\sim}$ is $E–c_1^{\sim}–c_2^{\sim}$, and this distance can also be written as $d_1^{\sim}+d_2^{\sim}–E$. The A&M definition identifies the share assigned to creditor $i$ as adding half the distance between $c_i^{\sim}$ and $d_i^{\sim}$ to the lower bound $c_i^{\sim}$; our new definition presents the share to creditor $i$ as subtracting half the distance between $c_i^{\sim}$ and $d_i^{\sim}$ from the upper bound $d_i^{\sim}$. Naturally, the two definitions are equivalent since the mid-point between $c_i^{\sim}$ and $d_i^{\sim}$ is identified in both alternatives. Thus, when viewed appropriately, each creditor may consider that she is awarded half the potential recovery or she is made to suffer half the total loss from bankruptcy; the amounts she receives are the same:

$$s_1–c_1^{\sim} = d_1^{\sim}–s_1 \text{ and } s_2–c_2^{\sim} = d_2^{\sim}–s_2.$$

As they share an equal amount of gain in the A&M approach, and they suffer an equal amount of loss in the alternative approach, we end up with the following:

$$s_1–c_1^{\sim} = d_1^{\sim}–s_1 = d_2^{\sim}–s_2 = s_2–c_2^{\sim}.$$

In other words, no matter which creditor we look at, and whether we view the partial repayment from the bankrupt estate as an individual *de facto* net gain or *de facto* net loss, under the CCC allocation principle they are all the same!

## 4. Gains and Losses Under the CCC bankruptcy allocations

That individual *de facto* net gain and individual *de facto* net loss are equal for a creditor and across creditors, whatever the size of the estate as long as there is bankruptcy, is an important distinguishing feature for CCC allocations and a fact not observed in the literature. Instead, the literature centers its attention on the related and perhaps more intuitive concepts of recovery and forfeiture. It stresses the sizes of nominal gains $s_i$ (the actual repayment without any qualification) and of nominal losses $d_i – s_i$ (the actual difference between the creditor's claim and the repayment) across creditors.[24] Table 4 below tabulates the relative magnitudes of these nominal concepts, where creditor 1's nominal claim is assumed to be less than creditor 2's nominal claim; it also presents the value of the *de facto* net gain $s_i –c_i^{\sim}$, which is identical across individual creditors and equals their common *de facto* net loss $d_i^{\sim}–s_i$ in the last column.

[24] A&M (1985, Footnote 18) states, "Specifically, whether we think of the outcome to Creditor $i$ as an award of $s_i$ or a loss of $d_i – s_i$." (The mathematical expressions have been adjusted to the notation used in this article.) This starts the trend in the literature to focus on nominal gain and the related nominal loss. Although it is correct in spirit, our analysis shows that it is more systematic to consider the outcome to creditor $i$ as a *de facto* gain $s_i – c_i^{\sim}$ or, equivalently, as a *de facto* loss $d_i^{\sim} – s_i$.

| | $s_1$ | $s_2$ | $d_1 - s_1$ | $d_2 - s_2$ | $s_1 \leftrightarrow s_2$ | $d_1 - s_1 \leftrightarrow d_2 - s_2$ | $s_i - c_i^{\sim} = d_i^{\sim} - s_i$ |
|---|---|---|---|---|---|---|---|
| $E \le d_1 \le d_2$ | $E/2$ | $E/2$ | $d_1 - E/2$ | $d_2 - E/2$ | $s_1 = s_2$ | $d_1 - s_1 \le d_2 - s_2$ | $E/2$ |
| $d_1 \le E \le d_2$ | $d_1/2$ | $E - d_1/2$ | $d_1/2$ | $d_2 - (E - d_1/2)$ | $s_1 \le s_2$ | $d_1 - s_1 \le d_2 - s_2$ | $d_1/2$ |
| $d_1 \le d_2 \le E$ | $(E + d_1 - d_2)/2$ | $(E + d_2 - d_1)/2$ | $(d_1 + d_2 - E)/2$ | $(d_1 + d_2 - E)/2$ | $s_1 \le s_2$ | $d_1 - s_1 = d_2 - s_2$ | $(d_1 + d_2 - E)/2$ |

***Table 4. Comparison of nominal gains, nominal losses, and de facto gains (same as de facto losses) under CCC allocations***

The first four columns in Table 4 present the nominal gains and the nominal losses for different ranges of values of the estate $E$. The next two comparison columns, along with the assumption that $d_1$ is less than $d_2$, indicate that nominal gains and nominal losses are order preserving. That is, under the CCC bankruptcy division, the nominal gain for a creditor with a lesser claim is less than (or equal to) the nominal gain for a creditor with a greater claim: $d_1 \le d_2 \Rightarrow s_1 \le s_2$. Likewise, a lesser creditor loses less than (or equal to) what a greater creditor would under the CCC bankruptcy division: $d_1 \le d_2 \Rightarrow d_1 - s_1 \le d_2 - s_2$.[25] Note that these relative nominal gains and nominal losses are order-preserving but not strictly order-preserving. In some regions, a creditor with a smaller claim receives the same nominal gain as a creditor with a larger claim.[26] In other regions, a creditor with a smaller claim incurs the same nominal loss as a creditor with a larger claim.[27]

It is not clear whether the equal nominal gain feature for small estates and the equal nominal loss feature for large estates inspired A&M (1985). In proving that a unique CCC solution exists for every bankruptcy problem, A&M (1985, p.200) describe how the allocation evolves when $E$ changes in two separation regions. When $E$ is less than half the total claim, attention is centered on the equality of the creditor shares with small $E$ and how an additional dollar gained is shared by the creditors as $E$ increases. When $E$ exceeds half the total claim, a mirror image of the previous process was used and focus is on the equality of individual loss when the estate is very large and how an additional dollar of total loss is borne by the creditors as $E$ decreases. In other words, individual gains $s_i$ are considered when $E$ is below half total claim while individual losses $d_i - s_i$ are contemplated when $E$ is above half total claim.[28]

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[25] Although we only illustrate results in the case of two creditors, applying the logic to any pair of creditors in the general case of $n$ creditors supports our conclusion. While they did not expand on the point as we do here, this order-preserving property was pointed out in A&M (1985, p.205).

[26] For example, when $E < d_1 < d_2$, $s_1 = s_2 = E/2$, and creditor 1 with a smaller claim receives the same nominal award as creditor 2 with a larger claim.

[27] For example, when $d_1 < d_2 = E$, $s_1 = d_1/2$, $s_2 = d_2 - d_1/2$, $d_1 - s_1 = d_1/2$, and $d_2 - s_2 = d_1/2$. Creditor 1 with a smaller claim suffers the same nominal loss as creditor 2 with a larger claim.

[28] After the full description of the CCC allocations, A&M (1985) also provide an alternative description in terms of increasing shares from additional $E$ for the case in which $E$ exceeds half the total claim.

The order preserving feature of both nominal gains and nominal net losses is in stark contrast with what is uncovered in the last section. Behind the CCC principle, with appropriate and intuitive restrictions on the basic variables, namely that a *de facto* concession $c_i^{\sim}$ cannot be negative and a *de facto* claim $d_i^{\sim}$ cannot exceed the estate value, the *de facto* net gains awarded to the lesser creditor and to the greater creditor are equal: $s_1–c_1^{\sim} = s_2–c_2^{\sim}$. Likewise, the *de facto* net losses suffered by the lesser and the greater creditors are also identical: $d_1^{\sim}–s_1 = d_2^{\sim}–s_2$. Further, the *de facto* net gain for any creditor $i$ equals the *de facto* net loss for any creditor $j$: $s_i – c_i^{\sim} = d_j^{\sim}–s_j$. These results come from the fact that under CCC division, a pair of creditors share the total *de facto* gains equally as well as suffer the total *de facto* loss equally. Thus, fundamentally, equal sharing is the name of the game for CCC allocations, whether we think in terms of gains or loss in the restricted *de facto* fashion. Looking a bit deeper, this means that one creditor counts just as much as another creditor, while attention is paid to the fact that outstanding claim exceeding the estate has no role in the repayment assignment.

In describing the CCC allocation explicitly, Aumann (2002) separates it in two parts. In the first part, "When the estate does not exceed half the sum of the claims, each woman gets the same amount, so long as this does not exceed half her claim." This part excludes scenarios in which $d_1 \leq d_2 \leq E$ hold, since $E$ would exceed half of the total claim otherwise. For the remaining two types of scenarios, both creditors always receive the same amount ($E/2$) if $E \leq d_1 \leq d_2$ (the first row in Table 4), in which case their award does not exceed half their individual claim. When $d_1 \leq E \leq d_2$ (the second row in Table 4), each creditor receives the same amount ($d_1/2$) only if $d_1 = E$. Combining these observations, each woman receives the same amount as long as $E \leq d_1$, or, as long as the estate is less than both claims. Thus, each creditor receives the same award only when $E \leq d_1$ but they receive identical *de facto* gains all the time. This is because when $E \leq d_1$ and thus $E \leq d_2$ as well, there are no concessions to speak of, and individual awards $s_i$ are equal as each award represents half of the total gain, which is the individual *de facto* net gain $s_i –c_i^{\sim}$.[29] When $E$ exceeds $d_1$, the positive but different concessions start kicking in and are added to half of the total gain, destroying equality of the awards if the creditor claims are unequal. As it happens, the award of the greater $d_2$-creditor is larger because the amount conceded to her ($E–d_1$) is larger than the amount conceded to the lesser $d_1$-creditor ($E–d_2$), while the *de facto* net gains remain equal as they represent the equal sharing part of total gains.

To complete describing the remaining part of the CCC allocation, Aumann (2002) states, "When the estate does exceed half the sum of the claims, the calculation is made in accordance with each woman's loss: the difference between her claim and what is actually paid out to her. The rule is that all the creditors lose the same amount, so long as none of them loses more than half her claim." Here, the requirement that the estate exceeds half the sum of the claims is incompatible with $E \leq d_1 \leq d_2$. In the other cases where $d_1 \leq d_2 \leq E$ (the third row in Table 4),

[29] This can be confirmed from comparing the first, second, and last cells on the first row in Table 4.

both creditors lose an equal amount (($d_1$+$d_2$−$E$)/2), and it is easy to confirm that each creditor loses less than her claim. In cases where $d_1 \le E \le d_2$ (the second row in Table 4), each creditor loses the same amount ($d_1$/2) as long as $E = d_2$. Thus, each creditor incurs the same loss only when $E \ge d_2$; but they bear identical *de facto* losses all the time. This is because when $E \ge d_2$ and thus $E \ge d_1$ as well, each creditor's nominal claim $d_i$ is her *de facto* claim $d_i^{\sim}$, making her nominal loss $d_i - s_i$ and her *de facto* loss $d_i^{\sim}-s_i$ one and the same;[30] each represents half of the *de facto* total loss $d_1^{\sim}+d_2^{\sim}-E$. When $E$ falls below $d_2$, the difference between the nominal claim and the *de facto* claim for the greater creditor is larger than the difference for the lesser creditor ($d_2-d_2^{\sim} > d_1-d_1^{\sim}$), destabilizing the equality between the nominal losses if creditor claims are different. As $E$ is below $d_2$, $d_2-d_2^{\sim} > d_1-d_1^{\sim}$ is equivalent to $(d_2-s_2)-(d_2^{\sim}-s_2) > (d_1-s_1)-(d_1^{\sim}-s_1)$. As $d_2^{\sim}-s_2$ and $d_1^{\sim}-s_1$ both equal half of the *de facto* total loss, this leads to $d_2-s_2 > d_1-s_1$, a larger loss for the greater $d_2$-creditor. Thus, the larger loss of the greater creditor comes from her relatively larger gap between money $d_2$ that is not all there and the value $E$ of the estate. ($d_2-E>d_1-E$ when $E<d_i$ implies that $d_2-d_2^{\sim} > d_1-d_1^{\sim}$). But *de facto* net losses remain identical since they represent the equal sharing part of the total loss.

To recap, while the literature stresses that CCC bankruptcy division requires equal sharing of total gains between any two creditors, it pays considerable attention to the relative gains $s_i$ and the relative losses $d_i - s_i$ across creditors. In particular, while the gains for the two creditors are equal at times and their losses are equal at other times, equalities only hold *some of the time*. Further, the gain for a creditor with a lesser claim is no greater than the gain for a creditor with a greater claim, and the loss for this creditor with a lesser claim is also no greater than the loss for a greater creditor. The fact that the gains and losses are order-preserving with respect to the sizes of creditors' claims is comforting; after all, those with a greater claim ought to be repaid no less than those with a lesser claim. But this study points out that other important equalities do hold *all the time*. Namely, the equalities of *de facto* net gains $s_i - c_i^{\sim}$ and of *de facto* net losses $d_i^{\sim}-s_i$ should be brought to bear instead of the gains and losses in general. The fact that equal-sharing of the total gains and equal-sharing of the total loss are equivalent in the refined *de facto* sense is rather surprising. Importantly, it provides us with a deeper understanding of the fundamental requirement behind the CCC principle.

## 5. The CCC Allocations for the Building-Block Case of Two Creditors

While the equal sharing principle of contested claim is straightforward, to implement and find the CCC division may not be straightforward, especially when it involves more than two creditors.[31] As noted earlier, in the proof for the unique existence of a CCC division, A&M

[30] This can be confirmed from comparing the third, fourth, and last cells on the last row in Table 4.

[31] Aumann (2002, p.6) notes, "Indeed, we are not dealing here with a method but, rather, with a condition. Given a certain division, one may check whether or not that division is CG-consistent. However, it may not be clear, at the

(1985) characterize the bankruptcy division, but they do not provide a detailed discussion of how this characterization comes about. When there are multiple creditors, any subset of creditors, including any two creditors, must follow the equal sharing principle of contested claim to split their award sum. This consistency requirement suggests some recursive steps are involved, and ultimately the bankruptcy division of two creditors underlie the division for any number of creditors. Tables of CCC divisions have not been readily available even for a small number of creditors, but would further our understanding.[32] To that end, this section characterizes the divisions for any two creditors, the building blocks for all to come. The next section investigates the divisions for three creditors.

The CCC divisions for two creditors were presented earlier in Tables 2 and 3, with the goal of confirming that two alternative ways of defining CCC divisions are equivalent. To identify the precise CCC allocation, given the claims of the creditors and the estate, Table 5 provides a useful alternative. It pays more attention to the boundaries where different types of divisions are called for across boundaries, and the boundary columns are highlighted. Strictly speaking, when the estate value is half the sum of the claims, it is not a boundary case. But since the literature stresses the difference in the estate allocation when the estate is above or below half the total claim, it is treated as a boundary case as well. To separate the types of divisions from one scenario to the next, it is convenient to redefine the estate by using the claims and additional parameters. The calculation of the shares assigned to the two creditors is straightforward. Appendix I presents the shares in a natural way, depending on the claims and the value of the estate $E$; they are then rewritten to depend on the claims and other parameters. This second presentation in Table 5 makes it easier to find the precise CCC allocation given fixed values for the claims and the estate; all parameters $\alpha$, $\beta$, $\beta'$, and $\alpha'$ are positive, with upper bounds clearly specified.

outset, how one arrives at a CG division." Note that "CG," Contested Garment in Aumann, has been renamed "CC," Contested Claim, in this article.

[32] Kaminski (2000) provides an ingenious way to visualize how an increase in the amount of a bankrupt estate should be divided among all creditors. But the approach suffers from the lack of explicit connection between the mathematical solution and the visual solution. For example, with a visual solution for two creditors, one needs to compute backward how $E$ should be related to $d_1$ and $d_2$. Going the other way, even in the case of three creditors, one may have to test a few visual solutions in order to find the correct one that leads to the mathematical form of the CCC division. This section organizes and lists all the CCC divisions as functions of the estate with fixed claims, but we did not enlist the help of visual solutions.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **Claim** \ $\boldsymbol{E}$ | $\mathbf{2\alpha}$ ($\alpha \leq d_1/2$) | $\boldsymbol{d_1}$ | $\boldsymbol{d_1+\beta}$ ($\beta \leq (d_2-d_1)/2$) | $\boldsymbol{(d_1+d_2)/2}$ | $\boldsymbol{(d_1+d_2)/2+\beta'}$ ($\beta' \leq (d_2-d_1)/2$) | $\boldsymbol{d_2}$ | $\boldsymbol{d_2+2\alpha'}$ ($\alpha' \leq d_1/2$) | $\boldsymbol{d_1+d_2}$ |
| $\boldsymbol{d_1}$ | $\alpha$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2+\alpha'$ | $d_1$ |
| $\boldsymbol{d_2}$ | $\alpha$ | $d_1/2$ | $d_1/2+\beta$ | $d_2/2$ | $d_2/2+\beta'$ | $d_2-d_1/2$ | $d_2-d_1/2+\alpha'$ | $d_2$ |

***Table 5. 2-Creditor Bankruptcy Shares Under Principle of Equal Division of Contested Claim ($d_1<d_2$)***

Table 5 confirms that as long as the estate is less than or equal to the smaller claim $d_1$ (columns 1 and 2, before crossing the first boundary), there is equal sharing of $E$ between the two creditors. As the estate increases beyond $d_1$ but less than $d_2$, the share assigned to the lesser creditor remains at $d_1/2$, and any extra increase in the value of the estate goes to repay the greater creditor. In this region where $d_1 < E < d_2$, the *de facto* concession offered to creditor 1 is zero, as the estate has not caught up with $d_2$ yet and no concession is offered, but the *de facto* concession given to creditor 2, $E-d_1$, increases as the estate $E$ increases ($\beta$ or $\beta'$ increases in Table 5). Meanwhile, the *de facto* total gain to be equally shared by both creditors holds still at $E-(E-d_1) = d_1$. This is why the share given to creditor 1 stays put at $d_1/2$, while the share given to creditor 2 keeps on increasing; the increase for creditor 2 is purely due to the higher conceded amount offered to him. Finally, note that Table 5 shows that when the bankrupt estate value increases beyond $d_2$ (column 7, beyond the second boundary), both creditors share the *extra* amount of the estate beyond $d_2$ equally. In this region where $d_1 \leq d_2 \leq E$, the amount to be shared by the two creditors is $d_1+d_2-E$,[33] which takes the form of $d_1+d_2-(d_2+2\alpha')=d_1-2\alpha'$. Adding half of this amount to the concessions due to creditors 1 and 2, $2\alpha'$ and $d_2-d_1+2\alpha'$, respectively, results in the shares shown in column 7. This column also implicitly confirms that both creditors' losses are equal. This is because when $d_1 < d_2 < E$, the *de facto* total loss and the nominal total loss are the same and take the value of $d_1+d_2-E = d_1+d_2-(d_2+2\alpha') = d_1-2\alpha'$.[34] This total loss, which of course equals the total gain shown a little earlier, is equally shared by the two creditors, leaving each creditor with a loss of $d_1/2-\alpha'$.

One final note not observed before is that when the value of the estate is half the total claim, each creditor receives half her claim; it turns out that this exceptional case is the only case where CCC repayments to the creditors also satisfy the proportionality principle.[35] The presentation of

[33] This is shown in the last row in Table 2.

[34] Alternatively, $s_2-s_1 = d_2^{\sim}-d_1^{\sim} = d_2-d_1$. The first equality comes from definitional manipulation, and the second equality holds because $d_1 \leq d_2 \leq E$; both properties can be observed in Table 3. Rewriting, we have $s_2-d_2 = s_1-d_1$.

[35] For the proof see Fon (2015*a*).

shares in Table 5 is limited to two creditors and the bankruptcy allocation mechanism should be extended to more creditors. To this we now turn in the next section.

## 6. The CCC Allocations for Three Creditors and the Nathan Examples

Following the full description of how the bankrupt estate is divided between two creditors under equal sharing of contested claim, we now concentrate on the division patterns for the case of three creditors for any value of estate $E$ with fixed claims $d_1$, $d_2$, and $d_3$. The proof in Appendix II shows explicitly how the CCC allocation for three creditors is found by consistently applying the equal division of contested claim to any two creditors, making use of results found in the 2-creditor case. This proof illustrates clearly the recursive structure behind the division patterns for any number of creditors.[36] Table 6 below duplicates the results in Table A2 of Appendix II, with highlights added to boundary cases in columns 2, 4, 8, and 10.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **E** / **Claim** | **3α,** ($\alpha \le d_1/2$) | **$3d_1/2$** | **$3d_1/2+2\beta$,** ($\beta \le (d_2-d_1)/2$) | **$d_1/2+d_2$** | **$d_1/2+d_2+\gamma$,** ($\gamma \le (d_3-d_2)/2$) | **$(d_1+d_2+d_3)/2$** | **$(d_1+d_2+d_3)/2+\gamma'$,** ($\gamma' \le (d_3-d_2)/2$) | **$d_3+d_1/2$** | **$d_3+d_1/2+2\beta'$,** ($\beta' \le (d_2-d_1)/2$) | **$d_2+d_3-d_1/2$** | **$d_2+d_3-d_1/2+3\alpha'$,** ($\alpha' \le d_1/2$) | **$d_1+d_2+d_3$** |
| $d_1$ | α | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2+\alpha'$ | $d_1$ |
| $d_2$ | α | $d_1/2$ | $d_1/2+\beta$ | $d_2/2$ | $d_2/2$ | $d_2/2$ | $d_2/2$ | $d_2/2$ | $d_2/2+\beta'$ | $d_2-d_1/2$ | $d_2-d_1/2+\alpha'$ | $d_2$ |
| $d_3$ | α | $d_1/2$ | $d_1/2+\beta$ | $d_2/2$ | $d_2/2+\gamma$ | $d_3/2$ | $d_3/2+\gamma'$ | $d_3-d_2/2$ | $d_3-d_2/2+\beta'$ | $d_3-d_1/2$ | $d_3-d_1/2+\alpha'$ | $d_3$ |

***Table 6. 3-Creditor Shares Under CCC Principle*** **($d_1<d_2<d_3$)**

With three creditors, the number of boundary cases expands to four from two in the case of two creditors. Observe that whenever the estate is no greater than $3d_1/2$ (columns 1 and 2), all creditors share the estate equally. Between the first and second boundaries (columns 3 and 4) in which $3d_1/2 < E \le d_1/2+d_2$, the share of creditor 1 stays constant at $d_1/2$, while creditors 2 and 3 split the amount of $E$ beyond $3d_1/2$. The higher shares for creditors 2 and 3 come from the amounts conceded by creditor 1 to each of them (under a two person setting). Likewise, between the second and third boundaries (columns 5 to 8), whenever a creditor's share is higher than another creditor, the difference comes from the concession offered to the former by the latter. Between the third and fourth boundaries (column 9), the sharing characteristics between

[36] As alluded to earlier, in their proof of the existence of a consistent solution to a bankruptcy problem, A&M (1985) characterize the CCC solution explicitly, but the characterization does not clearly illustrate the recursive nature and the consistency requirement involved.

creditors 1 and 2 and between 1 and 3 are the familiar one in the last scenario, where the joint sum lies between the pair of creditor claims. However, the joint award for creditors 2 and 3, $d_3+2\beta'$, exceeds both claims $d_3$ and $d_2$, resembling the relations between the estate $d_2+2\alpha'$ and claims $d_2$ and $d_1$ in column 7 of Table 5. There we found that creditors 1 and 2 equally share any extra amount beyond $d_2$, implying that now creditors 2 and 3 share any extra amount beyond $d_3$, or $2\beta'$. This is shown in column 9. Finally, beyond the fourth boundary (column 11), all three creditors incur the same loss of $d_1/2–\alpha'$.

Comparing Tables 5 and 6, a few interesting properties emerge. First, the first two rows in Table 5 are incorporated in Table 6, where the divisions for creditors 1 and 2 remain the same. This is to be expected, since consistency in CCC allocations means that taking creditors 1 and 2 as a subgroup, their allocation has to follow the basic principle of equal division of contested claim incorporated in Table 5. This is the result of the recursive nature of the problem mentioned earlier. Second, in terms of the smallest claim $d_1$, the length of the interval for equal gains increases as the number of creditors against the estate increases. In particular, the length of the equal-gain interval for two creditors is $d_1$, the length for three creditors is $3d_1/2$, and the length for $n$ creditors is $nd_1/2$.[37] On the other hand, in terms of total claim against the estate, the interval of *unequal* gains grows as the number of creditors grows. This is to be expected, since the requirement of equal shares for any two creditors, which implies equal shares for all creditors, becomes harder to fulfill as the number of creditors grow. Likewise, the behavior for equal losses follows a similar pattern. The length of the equal-loss interval for two creditors is $(d_1+d_2)–d_2=d_1$, for three creditors is $(d_1+d_2+d_3)–(d_2+d_3–d_1/2)=3d_1/2$, and for $n$ creditors is $nd_1/2$. Thus, in terms of the smallest claim $d_1$, the length of the interval for equal losses increases as the number of creditors against the estate increases. But in terms of total claim against the estate, the interval of unequal losses grows when there are more and more creditors. Here, readers are reminded that behind the CCC allocations, the *de facto* net gain and the *de facto* net loss are always equal across all creditors.

Next, we turn to the historically important Talmudic examples, where the claims of the creditors are fixed, but the estate value changes from case to case. Taking the general claims in Table 6 to be the specific values used in the Nathan examples, Table 7 incorporates the numerical results provided by Nathan, where $E$ equals 100, 200, and 300. In particular, when the estate is worth 100 (column 1), equal sharing for all is called for. When the value of the estate is 200 ($\beta$ = 25 in column 3), the least creditor receives 50, and each of the other two creditors receives an equal share of 75. When the estate is worth 300 (column 6), proportional division results.

[37] In general, let $d_1 \le d_2 \le \ldots \le d_n$. Table 5 shows that equal shares between creditor 1 and any other creditor $i$ ($s_1 = s_i \equiv s$) implies that $s \le d_1/2$. The total amount satisfying equal shares for $n$ creditors is then $ns$. Since $ns \le nd_1/2$, the interval of equal gains for all creditors is $nd_1/2$.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $E$ / $d_i$ | **3α** (α≤50) | **150** | **150+2β** (β≤50) | **250** | **250+γ** (γ≤50) | **300** | **300+γ′** (γ′≤50) | **350** | **350+2β′** (β′≤50) | **450** | **450+3α′** (α′≤50) | **600** |
| **100** | α | 50 | 50 | 50 | 50 | 50 | 50 | 50 | 50 | 50 | 50+α′ | 100 |
| **200** | α | 50 | 50+β | 100 | 100 | 100 | 100 | 100 | 100+β′ | 150 | 150+α′ | 200 |
| **300** | α | 50 | 50+β | 100 | 100+γ | 150 | 150+γ′ | 200 | 200+β′ | 250 | 250+α′ | 300 |

***Table 7. CCC Shares for Nathan's Example in the Talmud***

Table 7 extends different numerical tables given in the literature.[38] With these tabulations, we can calculate the division for any value of the estate with fixed claims 100, 200, and 300, thereby providing a complete and explicit answer to the two-thousand year old Talmudic question. For example, when the estate value is 400, column 9 (with β′=25) indicates that the 100-, 200-, and 300-claim creditors receive 50, 125, and 225 respectively.[39] The CCC allocations found so far clearly should be extended to any number of creditors. But doing so would be very tedious, conceptually not helpful, and beyond the scope of this article.

## 7. Conclusion

The CCC principle implicitly used in the Talmudic numerical examples is built on the basic case of 2 creditors. Aumann and Maschler defined it as the principle of equal sharing of contested claim. Each creditor is first awarded the amount conceded by the other creditor, the *de facto* concession; it is positive if the other creditor's claim is less than the estate's value and is zero otherwise. Each creditor then gets an additional amount which is half of the residual of the *de facto* concessions from the estate (the *de facto* contested claim). This article provides an alternative view of equal sharing of contested claim, namely, equal sharing of total loss, where total loss does not include any creditor's claim exceeding the estate (*de facto* total loss). It is shown that the two approaches are equivalent. This means that the principle involved splits everything down the middle, whether creditors view the estate as a gain or recognize the deficit of the estate from total claim as a loss.

The CCC principle has both ancient backing from the Talmud and modern support from a bargaining game solution. The CCC principle is equivalent to the game theoretical concept of nucleolus. The brilliance of the resulting CCC division actually lies heavily on the consistency

[38] Table 7 incorporates Table 6 in Aumann (2002), which provides the shares assigned to the three creditors as a function of the entire estate, by increments of 5. This table also generalizes table II, Appendix A, in Elishakoff and Begin-Drolet (2009), which provides each assigned share to a creditor as a function of the estate value, by increments of 10.

[39] Aumann (2002, p.9) used this specific value for the estate to describe the same CCC division as here.

requirement that for any pair (and more generally, any group) of creditors, their shares must satisfy the principle of equal sharing of contested claim. In the game theoretical setting to find nucleolus, a coalitional game corresponding to the bankruptcy problem with a set of creditors is formulated. The goal is to find an allocation that minimizes the worst inequality. Or, for each coalition of creditors, the dissatisfaction with the proposed allocation is contemplated. The nucleolus is the allocation that minimizes the maximum dissatisfaction. Put differently, it is impossible for any subset of creditors to get together and do better by themselves. Consistency requires this to be true regardless of the number of creditors in the subgroup; this method of dealing with fairness thus establishes stability among all creditors.

Other than characterizing a CCC allocation through equal sharing of *de facto* contested claim, the fact that a CCC division can also be described as fulfilling equal sharing of *de facto* total loss shows how thoroughly equal the treatment of the bankruptcy division is. Whether one views what is there or looks at what is missing, creditors always share equally. This makes a perfect match for *pari passu*, the Latin phrase meaning in equal step. The provision bearing this Latin name that was commonly found in the international debt instrument in the last century should incorporate equality every step of the way, gain or loss.[40]

The in-depth study of finding the CCC allocation through equal sharing of *de facto* total loss brings some issues to surface when compared to the proportional allocation commonly employed in bankruptcy situations. The CCC principle considers the part of the claim beyond the entire estate irrelevant, while the proportional principle takes the entire claim as relevant, including the portion that goes beyond the entire estate. The opportunity to obtain partial repayment and the burden to bear the short-fall are both split equally between the creditors in a CCC division; this demonstrates the fact that each creditor counts equally. This is very different from the proportional allocation treatment in which creditors do not count equally and only the amounts of the claim matter. The CCC principle takes each creditor as equally important and their individual claims relevant under appropriate restrictions, while proportional principle takes each dollar of the claim as equally important, independent of whom it belongs to and without restrictions. Thus, the CCC principle requires equal sharing of any contested claim (by both creditors), while the proportional principle requires sharing of the entire claim proportionally.[41]

In general, whether the CCC allocation or the proportional allocation better divides the bankrupt estate depends on what one views as the appropriate criteria. Since the CCC allocation embedded in the millennia old Talmudic lessons was not well understood, the balance tilt towards the use of the proportional allocation. Now that the underlying principles behind the

[40] Fon (2015*b*) argues this point in more detail.

[41] See Fon (2015*a*) for more in-depth analysis.

CCC allocation are better known, we should give strong consideration to CCC as an appropriate solution for a bankruptcy situation.

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## Appendix I: Equal Division of Contested Claim for Two Creditors

Without loss of generality, assume that the first claim is smaller than the second: $d_1 < d_2$. Given the bankrupt estate value $E$, the principle of equal division of contested claim specified by A&M (1985) provides the following shares to the two creditors, where $c_i^{\sim} = \max\{E–d_j,0\}$:

$$s_1 = c_1^{\sim}+[E–c_2^{\sim}–c_1^{\sim}]/2 = [E–c_2^{\sim}+c_1^{\sim}]/2, \text{ and } s_2 = c_2^{\sim}+[E–c_2^{\sim}–c_1^{\sim}]/2 = [E+c_2^{\sim}–c_1^{\sim}]/2.$$

While the general solutions for these divisions have been presented in Tables 2 and 3, this appendix provides more details. Special attention is paid to the behavior of the shares assigned to creditors as $E$ grows with fixed claims $d_1 < d_2$. Boundary cases separating different categories of divisions are added. In Table A1, Column 1 details the relation between the estate and the two claims. Column 2 presents the individual shares in terms of the estate $E$ and the claims; it also compares the creditors' shares as well as showing bounds for individual shares. In column 3, the estate is rewritten as a function of a claim or claims and a parameter, depending on the region, and creditor shares are presented in terms of the new expression.

| Given estate $E$ relative to the two claims $d_1$ and $d_2$ where $d_1 < d_2$ | Shares allocated to the creditors according to the equal sharing of contested claim principle | Share of Creditors expressed in terms of $d_1$, $d_2$ and a parameter |
|---|---|---|
| $E < d_1 < d_2$<br>($\Rightarrow$ $E < (d_1+d_2)/2$ ) | $s_1 = s_2 = E/2$<br>($\Rightarrow$ $s_1 < d_1/2$; $s_2 < d_1/2 < d_2/2$) | $s_1 = s_2 = \alpha$<br>where $E = 2\alpha$<br>($0 < \alpha < d_1/2$) |
| $E = d_1 < d_2$<br>($\Rightarrow$ $E = d_1 < (d_1+d_2)/2$ ) | $s_1 = s_2 = d_1/2$ | $s_1 = s_2 = d_1/2$<br>where $E = d_1$ |
| $d_1 < E < d_2$ and $E < (d_1+d_2)/2$ | $s_1 = d_1/2$, $s_2 = E – d_1/2$<br>($\Rightarrow$ $s_1 = d_1/2$; $d_1/2 < s_2 < d_2/2$) | $s_1 = d_1/2$, $s_2 = d_1/2 + \beta$<br>where $E = d_1 + \beta$<br>($0 < \beta < d_2/2 – d_1/2$) |
| $d_1 < E < d_2$ and $E = (d_1+d_2)/2$ | $s_1 = d_1/2$ and $s_2 = d_2/2$ | $s_1 = d_1/2$, $s_2 = d_2/2$<br>where $E = (d_1+d_2)/2$ |
| $d_1 < E < d_2$ and $(d_1+d_2)/2 < E$ | $s_1 = d_1/2$, $s_2 = E – d_1/2$<br>($\Rightarrow$ $s_1 = d_1/2$;<br>$d_2/2 < s_2 < d_2 – d_1/2$) | $s_1 = d_1/2$, $s_2 = d_2/2 + \beta'$<br>where $E = (d_1+d_2)/2 + \beta'$<br>($0 < \beta' < d_2/2 – d_1/2$) |
| $d_1 < E = d_2$ and $(d_1+d_2)/2 < E$ | $s_1 = d_1/2$ and $s_2 = d_2/2+(d_2–d_1)/2$ | $s_1 = d_1/2$, $s_2 = d_2 – d_1/2$<br>where $E = d_2$ |
| $d_1 < d_2 < E$ and $E < (d_1+d_2)$<br>($\Rightarrow$ $(d_1+d_2)/2 < E < (d_1+d_2)$) | $s_1 = (E+d_1–d_2)/2$, $s_2 = (E–d_1+d_2)/2$<br>($\Rightarrow$ $d_1/2 < s_1 < d_1$;<br>$d_2/2 < d_2 – d_1/2 < s_2 < d_2$) | $s_1 = d_1/2 + \alpha'$, $s_2 = d_2 – d_1/2+ \alpha'$<br>where $E = d_2 + 2\alpha'$<br>($0 < \alpha' < d_1/2$) |
| $d_1 < d_2 < E$ and $E = (d_1+d_2)$ | $s_1 = d_1$ and $s_2 = d_2$ | $s_1 = d_1$, $s_2 = d_2$<br>where $E = (d_1+d_2)$ |

***Table A1. Shares According to the Principle of Equal Division of Contested Claim* ($d_1<d_2$)**

Column 3 in Table A1 is built to help sort out which cases are relevant and make it easier to discern the bankruptcy solution by reformatting and combining the first two columns. In particular, $\alpha$ in the first row is defined to equal $E/2$, or, $E = 2\alpha$, which implies that $\alpha<d_1/2$. The second row is equivalent to the boundary case where $\alpha=d_1/2$. (Boundary cases are important to sort out where a certain numerical case falls among all possible scenarios.) In the third row, since $d_1/2<s_2$, $s_2$ is redefined to be $s_2=(d_1/2+\beta)$, where $\beta < d_2/2–d_1/2$ because $s_2<d_2/2$. Row four corresponds to the case in which $E$ is half the total claims and $\beta=d_2/2–d_1/2$. In the fifth row, since $d_2/2<s_2$, $s_2$ is redefined to be $s_2=(d_2/2+\beta')$, where $\beta'<d_2/2–d_1/2$ because $s_2<d_2–d_1/2$. The sixth row is the boundary case in which $\beta'=d_2/2–d_1/2$. In the seventh row, since $d_1/2< s_1$, $s_1$ is redefined to be $s_1=(d_1/2+\alpha')$, where $\alpha'< d_1/2$ because $s_1<d_1$. With the introduction of $\alpha'$, $s_2$ can be rewritten as $s_2 = (E–d_1+d_2)/2 = (E+d_1–d_2–2d_1+2d_2)/2 = s_1–d_1+d_2 = (d_1/2+\alpha')–d_1+d_2 = d_2–d_1/2+\alpha'$. The last row is included for completeness, as it is the limit of $\alpha'$ when $\alpha'$ approaches $d_1/2$; it is not related to the bankruptcy problem. The last column of Table A1 provides the easiest way to observe how different division cases are separated. It is regrouped in Table 5 in another tabular form for ease of extension to three creditors and beyond.

## Appendix II: CCC allocations for Three Creditors

In order to facilitate the proof of a 3-creditor CCC allocation, it is convenient to rewrite the restrictions on $E$ and the corresponding allocations for different scenarios found in the 2-creditor case. This is because consistency is required under the CCC principle. This means that the "estate" must be applied to the jointly awarded amount for any subset of two creditors repeatedly before it is clear what type of award solution is appropriate in the current setting. To do this, we first rewrite the solutions in Table A1 as follows.

1) [Rows 1–2] If $E \leq d_1 \leq d_2$, $s_1 = s_2 = \alpha$ ($\leq d_1/2$).
2) [Rows 3–6] If $d_1 \leq E \leq d_2$, $s_1 = d_1/2$, $s_2 = d_1/2+\beta$ ($\leq d_2 - d_1/2$); and $\beta \leq d_2 - d_1$.
   Note that we combine the allocations from rows 3 to 6 because the share given to the lesser creditor is constant at $d_1/2$. In doing so, the share to the greater creditor is written uniformly as $d_1/2+\beta$. Since $s_1+s_2 = E \leq d_2$, this means that $\beta \leq d_2 - d_1$.
3) [Rows 7–8] If $d_1 \leq d_2 \leq E$, $s_1 = d_1/2+\alpha'$ ($\geq d_1/2$), $s_2 = d_2 - d_1/2+\alpha'$ ($\geq d_2 - d_1/2$)

Rewrite the above three groups of solutions to facilitate our understanding of the basic assignment schedules for any two creditors $i$ and $j$. Let their joint allocation sum be $E^{ij}$. Presenting the above divisions in more general form in terms of $i$ and $j$, we have the following:
Case A. If $E^{ij} \leq d_i \leq d_j$, $s_i = s_j = \alpha$ ($\leq d_i/2$).
Case B. If $d_i \leq E^{ij} \leq d_j$, $s_i = d_i/2$, $s_j = d_i/2+\beta$ ($\leq d_j - d_i/2$), where $\beta \leq d_j - d_i$.
Case C. If $d_i \leq d_j \leq E^{ij}$, $s_i = d_i/2+\alpha'$ ($\geq d_i/2$), $s_j = d_j - d_i/2+\alpha'$ ($\geq d_j - d_i/2$).

In general, we assume that creditors are ranked according to the size of their claims; their claims satisfy $d_1 \leq d_2 \leq d_3$. (This ranking is for convenience.) From the results of the case of two creditors, we know that the allocations to the creditors are order preserving: $s_1 \leq s_2 \leq s_3$. The joint sum assigned to any two creditors $i$ and $j$ is $E^{ij} = s_i + s_j$. Thus $s_1 + s_2 \leq s_1 + s_3 \leq s_2 + s_3$ imply that $E^{12} \leq E^{13} \leq E^{23}$.

Without loss of generality, we prove the different bankruptcy allocations for three creditors assuming $d_1 < d_2 < d_3$. When we compile the results in Table A2, which follows the proof, we include all boundary cases. While the proof focuses on how the relation between the aggregate sum available to any two creditors and their claims characterizes individual repayments under bankruptcy, the converse proof from individual repayments to the aggregate sum-and-claims relation is straightforward. That is, the entries in Table A2 are complete characterizations under different values of the estate and claims on the estate.

I. Assume $E^{12} \leq d_1 < d_2$ ($< d_3$).
   Given $E^{12} \leq d_1$, case A implies that $s_1 = s_2 \equiv \alpha \leq d_1/2$. Assume that $d_2 \leq E^{23}$; then cases B and C imply that $s_2 \geq d_2/2$. This contradicts $s_2 \leq d_1/2$ since $d_1 < d_2$. Thus $E^{23} < d_2$ must hold in this case, and case A implies that $s_2 = s_3$, and $s_3 = \alpha$ also. To conclude:

(When $\alpha < d_1/2$) $s_1=s_2=s_3=\alpha$; $E=\Sigma s_i=3\alpha$. (Table A2 column 1)
(When $\alpha = d_1/2$) $s_1=s_2=s_3=d_1/2$; $E=\Sigma s_i=3d_1/2$. (Table A2 column 2)

II. Assume $d_1 < E^{12} \leq d_2$ and $E^{23} \leq d_2$.
When $d_1 < E^{12} \leq d_2$, case B implies that $s_1 = d_1/2$, $s_2 = d_1/2+\beta$, where $\beta\leq d_2–d_1$. With $E^{23} \leq d_2 < d_3$, case A implies that $s_2 = s_3$. Thus $s_3 = d_1/2+\beta$ as well, and $E^{23} \leq d_2$ implies that $d_1+2\beta\leq d_2$, or, $\beta\leq(d_2–d_1)/2$. (Note that $E^{12} = E^{13}$ in this case.) To conclude:
(When $\beta<(d_2–d_1)/2$) $s_1=d_1/2$, $s_2=s_3= d_1/2+\beta$; $E=\Sigma s_i=3d_1/2+2\beta$. (Table A2 column 3)
(When $\beta=(d_2–d_1)/2$) $s_1=d_1/2$, $s_2=s_3= d_2/2$; $E=\Sigma s_i=d_1/2+d_2$. (Table A2 column 4)

III. Assume $d_1 < E^{12} \leq d_2$ and $d_2 < E^{23} \leq d_3$.
Again $d_1 < E^{12} \leq d_2$ implies that $s_1=d_1/2$, $s_2=d_1/2+\beta$, where $\beta\leq d_2–d_1$. Likewise, $d_2 < E^{23} \leq d_3$ implies that $s_2=d_2/2$, $s_3=d_2/2+\gamma$, where $\gamma\leq d_3–d_2$. (Since the assigned amount to creditor 2 is unique, $d_1/2+\beta = d_2/2$ holds, and $\beta=(d_2–d_1)/2$ is implied.) What happens to the allocations depend on the value of $\gamma$.

a) (When $\gamma<(d_3–d_2)/2$) $s_1=d_1/2$, $s_2=d_2/2$, $s_3=d_2/2+\gamma$; $E=\Sigma s_i=d_1/2+d_2+\gamma$. (Table A2 column 5)
b) (When $\gamma=(d_3–d_2)/2$) $s_1=d_1/2$, $s_2=d_2/2$, $s_3=d_3/2$; $E=\Sigma s_i= (d_1+d_2+d_3)/2$. (Table A2 column 6)
c) When $(d_3–d_2)/2<\gamma< d_3–d_2$: Here $s_3=d_2/2+\gamma >d_3/2$. Define $s_3=d_3/2+\gamma'$. Since $E^{23} \leq d_3$, $s_2+s_3= d_2/2+d_3/2+\gamma'\leq d_3$ implies that $\gamma'\leq(d_3–d_2)/2$ holds. To conclude:
(When $\gamma'\leq(d_3–d_2)/2$) $s_1=d_1/2$, $s_2=d_2/2$, $s_3=d_3/2+\gamma'$; $E = (d_1+d_2+d_3)/2+\gamma'$. (Table A2 column 7)
d) (When $\gamma'=d_3–d_2$) $s_1=d_1/2$, $s_2=d_2/2$, $s_3=d_3–d_2/2$; $E = d_1/2+d_3$. (Table A2 column 8)

IV. Assume $d_1 < E^{12} \leq d_2$ and $d_3 < E^{23}$.
As usual, $d_1 < E^{12} \leq d_2$ implies that $s_1=d_1/2$, $s_2=d_1/2+\beta$, where $\beta\leq d_2–d_1$. ($d_2$ <) $d_3 < E^{23}$ and case C imply that $s_2=d_2/2+\beta'$, $s_3=d_3–d_2/2+\beta'$.

First, assume that ($d_1$ <) $d_3 < E^{13}$. Then case C implies that $s_1 = d_1/2+\beta''$, $s_3=d_3–d_1/2+\beta''$. Since $s_1=d_1/2$ also, $\beta''=0$; and $s_3=d_3–d_1/2$. This implies that $E^{13}= s_1+s_3 = d_3$, a contradiction to the assumption that $d_3 < E^{23}$.

Thus, ($d_1$ <) $E^{13} \leq d_3$ must hold. Case B implies that $s_1 = d_1/2$, $s_3 = d_1/2+\beta''$ where $\beta''\leq d_3–d_1$. The allocation amount for creditor 3 should be unique: $s_3=d_3–d_2/2+\beta'= d_1/2+\beta''$ implies that $\beta''= d_3–d_1/2–d_2/2+\beta'$. Together with $\beta''\leq d_3–d_1$, we have $\beta'\leq (d_2–d_1)/2$ must hold. (Note that the uniqueness of $s_2$ implies that $\beta=(d_2–d_1)/2+\beta'$, which further implies that $\beta\leq d_2–d_1$, consistent with what is required.) To conclude:
(When $\beta'<(d_2–d_1)/2$) $s_1=d_1/2$, $s_2=d_2/2+\beta'$, $s_3=d_3–d_2/2+\beta'$; $E= d_3+d_1/2+2\beta'$. (Table A2 col 9)
(When $\beta'=(d_2–d_1)/2$) $s_1=d_1/2$, $s_2= d_2–d_1/2$, $s_3=d_3–d_1/2$; $E=d_2+d_3–d_1/2$. (Table A2 col 10)

V. Assume $d_1 < d_2 < E^{12}$ $(< E^{13} < E^{23})$.

From case C, $d_1 < d_2 < E^{12}$ implies that $s_1 = d_1/2+\alpha'$ and $s_2 = d_2 - d_1/2+\alpha'$.

Now assume that $(d_1 <)\ E^{13} < d_3$. Then case B dictates that $s_1 = d_1/2$, $s_3 = d_1/2+\beta$, $\beta>0$. Under this assumption, uniqueness of $s_1$ implies that $\alpha' = 0$, and $s_1=d_1/2$ and $s_2=d_2-d_1/2$. Further assume that $d_2 < E^{23} < d_3$. Again case B provides that $s_2 = d_2/2$, $s_3 = d_2/2+\beta''$, $\beta''>0$. Uniqueness of $s_2$ then implies that $d_2-d_1/2 = d_2/2$ which further implies that $d_1 = d_2$. This is a contradiction. The remaining case requires the assumption $d_2 < d_3 < E^{23}$, where case C gives $s_2=d_2/2+\alpha''$ and $s_3=d_3-d_2/2+\alpha''$. Uniqueness of $s_2$ then implies that $\alpha''=(d_2-d_1)/2$, which further implies that $s_3=d_3-d_1/2$. This means that $E^{13}= s_1+s_3 = d_3$; this is also a contradiction. Combining the two cases considered, we conclude that $d_3 \le E^{13}$ must be true.

Thus, $(d_1 <)\ d_3 \le E^{13}$ must hold. Now case C provides that $s_1=d_1/2+\alpha''$ and $s_3=d_3-d_1/2+\alpha''$. Uniqueness of $s_1$ implies that $\alpha'' = \alpha'$ and the allocations are $s_1=d_1/2+\alpha'$, $s_2=d_2-d_1/2+\alpha'$, and $s_3=d_3-d_1/2+\alpha'$. The bankruptcy problem implies that the total allocation to all creditors is no larger than the total debt owed to all creditors: $s_1+s_2+s_3 \le d_1+d_2+d_3$. This requires that $\alpha' \le d_1/2$. (Strictly speaking, there is no bankruptcy problem when $\alpha'=d_1/2$.) To conclude:

(When $\alpha'<d_1/2$) $s_1=d_1/2+\alpha'$, $s_2=d_2-d_1/2+\alpha'$, $s_3=d_3-d_1/2+\alpha'$;$E=d_2+d_3-d_1/2+3\alpha'$. (Table A2 col 11)

(When $\alpha'=d_1/2$) $s_1=d_1$, $s_2=d_2$, and $s_3=d_3$; $E=d_1+d_2+d_3$. (Table A2 col 12)

We conclude that the CCC allocations for the case of three creditors are the following.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **E** / **Claim** | **$3\alpha$,** ($\alpha \le d_1/2$) | **$3d_1/2$** | **$3d_1/2+2\beta$,** ($\beta \le (d_2-d_1)/2$) | **$d_1/2+d_2$** | **$d_1/2+d_2+\gamma$,** ($\gamma \le (d_3-d_2)/2$) | **$(d_1+d_2+d_3)/2$** | **$(d_1+d_2+d_3)/2+\gamma'$,** ($\gamma' \le (d_3-d_2)/2$) | **$d_3+d_1/2$** | **$d_3+d_1/2+2\beta'$,** ($\beta' \le (d_2-d_1)/2$) | **$d_2+d_3-d_1/2$** | **$d_2+d_3-d_1/2+3\alpha'$,** ($\alpha' \le d_1/2$) | **$d_1+d_2+d_3$** |
| **$d_1$** | $\alpha$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2$ | $d_1/2+\alpha'$ | $d_1$ |
| **$d_2$** | $\alpha$ | $d_1/2$ | $d_1/2+\beta$ | $d_2/2$ | $d_2/2$ | $d_2/2$ | $d_2/2$ | $d_2/2$ | $d_2/2+\beta'$ | $d_2-d_1/2$ | $d_2-d_1/2+\alpha'$ | $d_2$ |
| **$d_3$** | $\alpha$ | $d_1/2$ | $d_1/2+\beta$ | $d_2/2$ | $d_2/2+\gamma$ | $d_3/2$ | $d_3/2+\gamma'$ | $d_3-d_2/2$ | $d_3-d_2/2+\beta'$ | $d_3-d_1/2$ | $d_3-d_1/2+\alpha'$ | $d_3$ |

***Table A2. 3-Creditor Shares Under the CCC Principle* ($d_1<d_2<d_3$)**