Relational Contracts, Financing Constraints and Social Welfare*

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Abstract

This paper analyzes the interaction between relational contracting and the quality of financial markets. I study a simple model in which a downstream firm (the buyer) sources components from an upstream firm (the supplier). The parties interact without the benefit of a formal contract and, due to imperfections in financial markets, the supplier has limited access to credit. The buyer is then required to cover a fraction of the investment cost. I characterize the whole set of efficient self-enforcing contracts and analyze how they are affected by the magnitude of the financing friction. If the supplier has strong bargaining power, less efficient financial markets may be beneficial to social welfare. If the buyer has strong bargaining power, on the other hand, less efficient financial markets are always welfare-reducing. The model also predicts that the productivity of the partnership increases with the length of the relationship. After a finite number of periods, however, the relationship "matures" and every efficient self-enforcing contract converges to a stationary agreement that maximizes social welfare among the class of all self-enforcing contracts. During the "transition phase", investment decisions are distorted, resulting in either under- or over-investment. Over time, the inefficiencies decrease and investment monotonically approaches its first-best level.

Keywords: Financing Constraints, Hold-up Problem, Imperfect Enforcement, Self-Enforcing Contracts.

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1 Introduction

When formal institutions cannot enforce contractual rights, firms resort to alternative methods, in particular to relational contracting. Under relational contracting, firms develop long-term relationships with their partners and use the ongoing nature of the relationship to enforce an implicit contract—provided that the firms anticipate a profit, the threat of dissolving the partnership acts as an incentive for both of them to honor their promises.

While many authors have documented the use of relational contracting, the extent to which it can help circumvent difficulties in formal contracting and thereby generate growth is not well understood. In China and Vietnam, for example, relational contracting has been able to sustain high growth rates despite inadequate legal systems. But many other countries, apparently similar in terms of both the quality of their legal systems and the widespread use of relational contracts, have done quite poorly.[1] Why is relational contracting able to sustain high growth rates in some countries but no others? In this paper I offer an answer to this question by showing that the ability of relational contracting to alleviate difficulties in formal contracting depends on the quality of formal financial market as well as the allocation of bargaining power between the firms.

To formally address this issue, I study an infinite-horizon model in which a buyer sources components from a supplier and contracts are not enforceable by the government or any other third party. At the beginning of every period, the buyer offers the supplier an initial transfer and promises ex post payments contingent upon the number of components delivered. If the offer is accepted, the supplier receives the initial payment and decides how many components to build. Once the components have been produced, at the end of the period, both firms decide whether to honor the contract. If they both do, the supplier delivers the components in exchange for the agreed transfer. If either firm breaches the contract, on the other hand, the supplier retains the components and the firms engage in negotiations, which result in the sharing of ex post gains from trade according to an exogenously determined rule. The fraction of the total revenue received by the supplier is fixed over time and represents his bargaining power.

To introduce financial markets into the model, I assume that the supplier has no initial wealth but is endowed with an asset that can be pledged as collateral to a bank. The supplier can, however, only borrow a fraction of the investment cost. The buyer, who has access to large amount of resources, must cover the rest through the initial payment. How much the

[1] For evidence on Vietnam and China, see McMillan and Woodruff (1999) and Allen, Qian, and Qian (2005). For evidence on the failure of relational contracting to sustain growth, see the work of Fafchamps (2004) on Africa.

supplier can borrow from the bank measures the level of frictions in financial markets and provides a simple way of capturing the degree of credit rationing in the economy.

Following MacLeod and Malcomson (1989), I model self-enforcing relational contracts as equilibrium outcomes of the repeated game. Using the recursive method developed by Abreu, Pearce, and Stacchetti (1990), I then characterize the whole set of Pareto efficient self-enforcing relational contracts and derive implications for the behavior of investment and transfers over time. The first main result shows that after a finite number of periods, *every* efficient self-enforcing relational contract converges to a stationary contract—a simple agreement in which the buyer offers the same compensation scheme and the supplier produces the same amount of components in every period. Thus, while there is a continuum of self-enforcing relational contracts, the investment level is uniquely determined in the long run. Since these stationary contracts also have the property of maximizing the firms' joint profits among the class of all self-enforcing contract, I call them *socially optimal contracts*.

The model makes three predictions regarding transfers once self-enforcing contracts reach the stationary phase. First, the initial transfer from buyer need not be large enough to cover the entire initial investment. In this case, the contract has an interesting interpretation: it resembles a *joint venture agreement or partnership* in which the buyer covers a fraction of the investment cost in exchange of debt and a fraction of the project's shares. Second, as the supplier's profits rise, they become less dependent on the profitability of the project. The supplier is therefore treated less as a partner and more like an external supplier, who simply charges the final price of the components in advance. Finally, the ex post transfers to the supplier may be negative, which can be thought of as a payback agreement. The buyer contributes a share of the investment cost larger than required by the supplier's financing constraint. In exchange, he demands compensation when market conditions turn out to be extremely bad.

I then study how the level of social welfare achieved by socially optimal contracts is affected by variations in the magnitude of the financing friction. I find that the relationship between the financing friction and the level of social welfare depends critically on the allocation of bargaining power. If the buyer has strong bargaining power, less efficient financial markets always reduce social welfare. By contrast, if the supplier has strong bargaining power, less efficient financial markets may increase social welfare. By showing that a certain degree of financing frictions can *complement* relational contracting, the model establishes a novel channel through which finance affects the real economy.

The intuition behind the previous result is as follows. Since the buyer has to make a larger initial payment, less efficient financial markets increase the supplier's short term payoffs from reneging on the contract, making it harder for the firms to sustain the original level of investment. But less efficient financial markets also affect the firms' long-term deviation payoffs—the firms' payoffs once the relationship is terminated. This generates a second effect whose direction depends on the allocation of bargaining power between the firms. If the buyer has strong bargaining power, the supplier's long-term deviation payoffs increase when financing conditions worsen, thus weakening incentives. In this case both effects work in the same direction and social welfare unambiguously decreases. If the supplier has strong bargaining power, on the other hand, both firms' long-term deviation payoffs decrease when financing conditions worsen, thus strengthening incentives to respect the agreement. Social welfare can then either decrease or increase, depending on which of the two effects dominates.

Finally, I analyze the behavior of investment during the "transition phase". I show that investment decisions are initially distorted, but these inefficiencies decrease over time and investment monotonically approaches its first best level. Depending on the magnitude of the financing friction and the value of the relationship, the initial distortions can result in either under- or over-investment. In particular, the possibility of over-investment arises because investment can be used by the supplier to simulate an up front payment to the buyer, which under some conditions can facilitate cooperation. It is worth noting that possibility of over-investment stems from the combination of contractual and financing frictions and not from agency problems between the manager of a firm and its shareholders. Thus, the model provides a complementary explanation to one of the most well-established stylized facts in the corporate finance literature: the tendency of managers to over-invest (see, e.g., Stein, 2003).

The rest of the paper is organized as follows. The next section discusses the related literature. Section 3 introduces the model. Sections 4 and 5 analyze contracts under perfect enforcement and spot transactions, respectively. Section 6 contains a formal definition of self-enforcing relational contracts and offers a recursive characterization. Section 7 studies efficient self-enforcing contracts without the financing constraint. Section 8 discusses the main properties of efficient self-enforcing relational contracts when financial markets are imperfect and analyzes how they are affected by the magnitude of the financing friction. All the proofs are contained in the Appendix.

2 Related Literature

In this section I describe the related literature and highlight how this paper contributes to the current body of work. The paper is closely related to a number of contributions analyzing how contractual frictions interact with other distortions. These papers also identify mechanisms through which a reduction in the latter can decrease social welfare. Baker, Gibbons, and Murphy (1994) and Kovrijnykh (forthcoming), for instance, analyze the interaction between formal and relational contracts. The authors show that an improvement in the formal enforcement technology can reduce social welfare by increasing the agents' payoffs in the equilibrium that serves as optimal punishment to enforce the implicit agreement. Kvaløy and Olsen (2009), on the other hand, consider a repeated principal-agent model in which the verifiability of the agent's action is endogenously determined by the principal's effort. The authors then study how variations in the discount factor and the verification cost affect the properties of relational contracts. Unlike these papers, I consider the interaction between contractual frictions and imperfections in credit markets and study efficient relational contracting under different allocations of bargaining power.

Thomas and Worrall (2010) and Fong and Li (2012) are two recent papers that combine financial and enforcement frictions. Thomas and Worrall (2010) study a partnership game in which output depends on the joint effort, both agents are liquidity constrained, and the output must be shared at the end of each period. The authors show that at an early stage of the relationship, effort may be above its efficient level, which is consistent with my results. In the context of a repeated principal-agent model with moral hazard and limited liability, Fong and Li (2012) explore the properties of efficient relational contracts in terms of employment characteristics (probability of termination, wages, and sensitivity of pay to performance). The main difference between my work and these two papers is that I allow the liquidity-constrained agent to borrow from financial markets. The amount of borrowing is, however, limited by the level of frictions in financial markets, which allows me to analyze how efficient relational contracts depend on the quality of credit markets.

The idea that better financial markets can decrease the incentives to comply with an implicit agreement is also explored by Bulow and Rogoff (1989). These authors analyze the repayment incentives of a small country borrowing from international capital markets to smooth consumption under two alternative scenarios. If the borrowing country is completely isolated from financial markets after a default, a positive level of debt can be sustained in equilibrium. In contrast, if irrespective of his past behavior the country can purchase an insurance contract that delivers payments in low output states exactly like borrowing would, then no lending can be sustained. Thus, having access to better financial instruments destroys all equilibria with positive debt and decreases welfare. My approach differs from theirs, however, in that I allow for multiple levels of financial frictions and different levels of bargaining power between the parties. In section 8.3 I show that these differences lead to contrasting results regarding the impact of imperfections in credit markets on social welfare.

My analysis of the role played by the agents' bargaining power is related to the work of Genicot and Ray (2006). In a credit market model with self-enforcing credit constraints, the authors study how a change in the outside option of a potential defaulter affects the terms of the contract and how this depends on the allocation of bargaining power. Their finding that a decrease in the borrower's outside option can increase his utility is consistent with my results. In my model, however, the value of the outside option is endogenously determined by the level of the financing friction, which also has a direct impact on the structure of self-enforcing contracts.

The paper is also related to the literature on dynamic financial contracts. Clementi and Hopenhayn (2006) and DeMarzo and Fishman (2007) explore the implications of incentive problems caused by private information about cash flows on investment dynamics. In contrast to my model, both papers assume that agents can commit to a long-term contract. Closer to my work is Albuquerque and Hopenhayn (2004), who consider a lending model with endogenous financing constraints in which the borrower's outside option depends on the realization of an observable and persistent shock. In their model, as opposed to mine, the lender has perfect commitment power and the firm is eventually either liquidated or investment converges to the efficient level.

Finally, there is a small literature in supply chain management analyzing the impact of weak enforcement on capacity and pricing decision (see, e.g., Plambeck and Taylor (2006), Taylor and Plambeck (2007a,b), and Ren, Cohen, Ho, and Terwiesch (2010)). These papers share with mine the focus on the use of informal agreements that are sustained by repeated interaction. They are, however, typically interested in the performance of "simple" contracts relative to the first-best outcome. Thus, the set of available agreements is exogenously restricted. Exceptions are Plambeck and Taylor (2006) and Taylor and Plambeck (2007a), who characterize fully optimal self-enforcing agreements. In contrast to my model, in both papers, firms are liquidity unconstrained and contracts take a simple stationary form (see Levin (2003)).

3 Model Setup

Consider the following environment. Time is discrete and indexed by $t \in \{1, 2, 3, ..., \infty\}$. The economy is composed of two infinitely-lived agents: a downstream firm (the "buyer") and an upstream firm (the "supplier"). Both firms are risk neutral, discount future profits using the same discount factor $\delta \in (0, 1)$, and seek to maximize the present value of their own expected profits.

3.1 Technology

At the beginning of each period, the buyer sources a perishable component from the supplier. The buyer then uses the components to produce final goods and sells them to consumers in a final market. The buyer's only cost of production is that of the components. Units are normalized so that one unit of the component is required to produce one unit of the final good. The revenue generated from selling k units of the final good in period t is given by $R(k, \theta_t)$, where θ_t denotes the realization of a discrete random variable with support $\Theta := \{1, 2, ..., n\}$. One can think of θ_t as reflecting any factor affecting the profitability of the final product. For example, the availability of similar goods at the time the product is released, the size of the market, or the consumers' willingness to pay. The parameter θ_t is observed by both firms, *i.i.d.* over time, and takes the value $\theta \in \Theta$ with probability f_θ. Assume that for all $\theta \in \Theta$, the revenue function $R(\cdot, \theta) : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave, continuously differentiable, and satisfies $R(0, \theta) = 0$, $\lim_{k\to\infty} R_k(\cdot, \theta) = 0$, and $\lim_{k\to 0} R_k(\cdot, \theta) = \infty$. In addition, assume that a higher value of θ indicates more favorable market conditions, i.e., for all $k \in \mathbb{R}_+$, the function $R(k, \cdot) : \Theta \to \mathbb{R}_+$ is strictly increasing.

Due to a long acquisition lead time, the supplier needs to make capacity decisions before the market parameter θ is realized, where capacity refers to all resources (e.g., labor, raw material, production facility, etc.) needed to produce the component. Let $c > 0$ denote the supplier's per unit cost of capacity and assume, for simplicity, that capacity can be instantly converted into the component at zero cost. Thus, an alternative interpretation of the model is that the supplier builds up inventory to satisfy the buyer's order.

3.2 Financing Constraints

A crucial feature of the model is that the buyer and the supplier face different financial conditions. On the one hand, the buyer has a large amount of resources and perfect access to credit markets. The supplier, on the other hand, has no initial wealth but is endowed with an asset A that can be pledged as collateral to a bank. Assume that at the beginning of each period, the supplier approaches the bank and using the asset as collateral, asks for a loan to produce k units of the component. Provided the bank and the supplier are in "good standing", the bank will agree to finance at most a fraction $1 - \phi$ of the total cost of investment (i.e., $(1 - \phi) ck$). The rest, at least a fraction ϕ, must be covered by the buyer.[2]

[2]This is consistent with the idea that because the buyer may have a comparative advantage in monitoring and exploiting informal means of avoiding opportunistic behavior by the supplier, his contribution to investment acts as a positive signal for the bank, thus decreasing credit rationing; see, e.g., Biais and Gollier (1997), Demirguc-Kunt and Maksimovic (2001), Burkart and Ellingsen (2004), and Frank and Maksimovic

If, after obtaining the credit, the supplier reneges on his promise and chooses a different level of investment or refuses to repay the loan, the bank will seize the asset and will exclude the supplier from any future lending.

The parameter ϕ is key in the model. When $\phi = 0$, financial markets are frictionless and the supplier can finance the entire investment cost using credit from the bank. On the other hand, when $\phi = 1$, the supplier does not have access to credit and the buyer must cover the entire investment cost. The value of ϕ then measures the level of frictions in financial markets and provides a simple way of capturing the degree of credit rationing in the economy.

To direct focus to the relationship between the buyer and the supplier, I make the following simplifying assumptions. First, I assume that the supplier's valuation of the asset A is large enough to ensure that the he never finds it advantageous to deviate from the promised level of k or to refuse to pay his debt to the bank. Second, I assume that the bank is risk neutral, perfectly competitive (needs to only break even), and discount cash flows at the same rate as the firms. The bank is therefore indifferent about the specific timing of the repayment, as long as its discounted expected value equals the size of the loan. Third, I assume that if the supplier's profits are not large enough to fully repay the bank, the latter can sell a part or the whole asset to recover the loan. Of course, repayment to the bank would always be possible in equilibrium. This assumption, however, ensures that repayment is possible even after a deviation from equilibrium actions. Finally, I assume that the supplier does not save. Any transfer received from the buyer (not used to repay the bank) will therefore be immediately consumed.[3]

Note that the previous assumptions imply that the bank behaves as a passive agent, always willing to finance a fixed fraction of the total cost of investment, regardless of the history of play. Moreover, they amount to assuming that, in each period, the supplier himself can cover at most a fraction $1 - \phi$ of the cost of investment. Of course, the above description of the interaction between the bank and the supplier is not meant to be realistic. Rather, it is a simple way of introducing the idea that credit market frictions induce credit rationing.

To summarize, the supplier's financial condition and the frictions in credit markets imply that the supplier's ability to make the initial investment is subject to the following constraint:

$$ck \leq l + (1 - \phi)\, ck,$$

where l denotes the initial payment from the buyer.

(2005).

[3]This last assumption is consistent with empirical evidence by Johnson, McMillan, and Woodruff (2002), who find that weak property rights discourage firms from reinvesting profits out of fear of expropriation.

3.3 Timing and Contracts

The timing of events within each period is as follows.[4] At the beginning of date t, before the market parameter θ is realized, the buyer proposes a contract or compensation scheme to the supplier. The offer specifies an initial transfer l_t and ex post payments $T_t : \mathbb{R}_+ \times \Theta \to \mathbb{R}_+$ contingent upon the number of components delivered by the supplier and the realization of θ. Contracts are not enforceable by the government or any other third party. Therefore, at any point in time, either firm can renege on the agreement. This implies that the contingent payment T_t is merely a promise and can thus be repudiated.

After receiving the buyer's offer, the supplier decides whether to accept it. Let $p_t \in \{0.1\}$ denote the supplier's decision. If the supplier rejects the buyer's original offer ($p_t = 0$), the firms temporarily revert to the equilibrium of the game under spot transactions, which I will described in section 5. Alternatively, if the contract is accepted ($p_t = 1$), the supplier receives the initial payment l_t, obtains the loan from the bank, and chooses how many units of capacity to build. At the end of the period, after capacity is built and the demand parameter θ_t is revealed, the buyer offers a payment of w_t to the supplier in exchange for the components. As mentioned earlier, w_t need not coincide with the transfer $T(\theta)$ specified in the original contract. Furthermore, even when it does, the supplier is free to reject the payment if doing so increases his profits. To complete the mode, it is then necessary to specify the actions available to the parties in the event of disagreement about ex post payments. Let $d_t \in \{0, 1\}$ denote the supplier's decision regarding w_t. Following Grossman and Hart (1986), if the ex post payment is rejected ($d_t = 0$), firms bargain over the terms of a new agreement and the outcome of such negotiation is given by the generalized Nash bargaining solution.[5] Bargaining then results on the efficient use of the components (i.e., the buyer acquires all units) and on the sharing of ex post gains from trade, with the seller receiving a fraction $\beta^s \in (0, 1)$ of the total revenue. The parameter β^s, the supplier's bargaining power, is exogenously determined and constant over time.[6] Figure 1 summarizes the sequence of events within each period.

[4]Though not explicitly stated, I assume that at the beginning of each period, before any action is taken, both firms observe the realization of a public randomization device. This assumption guarantees the convexity of the set of equilibrium payoffs.

[5]It is possible to explicitly model the bargaining procedure as a game in which firms alternate in making offers until one of them is accepted. In the unique equilibrium of such game, an agreement is reached immediately and the division of surplus depends on which firm makes the first offer, the timing between offers, and the parties' beliefs concerning the risk of a break-down of negotiations; see Rubinstein (1982) and Binmore, Rubinstein, and Wolinsky (1986) for further discussion.

[6]To properly define a Nash bargaining model, it is also necessary to define the firms' threat points. I follow Binmore, Rubinstein, and Wolinsky (1986) and assume that they are given by the profits gained during the negotiations (i.e. zero for both firm).

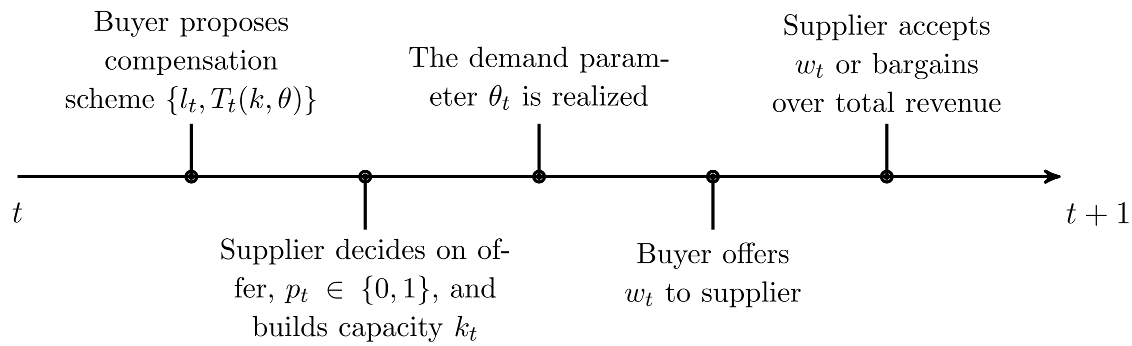


Figure 1: Timeline

4 Complete Contracts

As a first step and for future reference, I start by analyzing the properties of efficient contracts in the absence of commitment problems. Before turning to this task, note that when the supplier builds k units of capacity, joint expected profits are given by

$$\pi^J(k) = -ck + \mathbb{E}[R(k,\theta)], \tag{4.1}$$

where $\mathbb{E}$ is the expectation operator with respect to the demand parameter θ. Given the assumptions on the revenue function, the *first-best or efficient level of capacity*, denoted by k^{FB}, is the unique solution to the following first-order condition: $-c + \mathbb{E}\left[R_k\left(k^{FB},\theta\right)\right] = 0$.

Now suppose for a moment that both capacity and payments are fully enforceable so that firms cannot renege on their promises. The next proposition shows that financial frictions alone do not suffice to generate investment inefficiencies. This will stand in contrast to the results of the following sections, in which contract enforcement is weak and firms must rely on relational (self-enforced) contracts.

Proposition 4.1 (Complete Contracts) *Suppose that both firms can credibly commit to any contract. Then, in any Pareto efficient contract and regardless of the magnitude of the financing friction (i.e., the value of ϕ), investment is efficient in each period, i.e., $k_t = k^{FB}$ for all $t \geq 1$.*

Thus, absent any enforcement problem, the financing friction has no effect on social welfare. The intuition behind Proposition 4.1 is straightforward. By forcing the buyer to make an initial payment to the supplier, the financing constraint affects the timing of transfers. Because both firms are risk neutral and contracts can be enforced, however, payments can be manipulated to satisfy the supplier's liquidity and financing restrictions without affecting the firms' profits or actions. As a consequence, when contract enforcement is not a concern, Pareto efficient contracts deliver a first-best outcome and exhaust all gains from investment.

5 Spot Transactions

When the buyer and the supplier meet only once or do not use history dependent strategies, they engage in spot transactions. In this section, I calculate the equilibrium of the game under these circumstances. As I show in the next section, the equilibrium under spot transactions will be used as a threat against deviation from equilibrium strategies in the repeated game. The fact that firms' profits under this equilibrium are endogenous—affected by the parameter of the model, as shown by Proposition 5.3 below—will then turn out to be key when analyzing the impact of an increase in the financing friction on the properties of efficient self-enforcing contracts.

Since ex post transfers are not contractible, firms will split ex post gains from trade according to the generalized Nash bargaining solution. As a consequence, anticipating a share β^s of the total revenue and given an initial payment of l from the buyer, the supplier's expected profits are

$$\pi^s_{spot}(k,l) := l - ck + \beta^s \mathbb{E}\left[R(k,\theta)\right].$$

The supplier's problem is then to choose a level of capacity so as to maximize $\pi^S_{spot}(k,l)$ subject to the liquidity constraint $\phi ck \leq l$. It is useful to note that if the supplier could raise enough resources to fully pay the investment cost (i.e., $\phi = 0$), his optimal decision would be given by

$$\underline{k}^*_u := \arg\max_{k\geq 0} \pi^S_{spot}(k,l). \tag{5.1}$$

Moreover, because $\underline{k}^*_u$ is independent of l, no initial payment would take place. Thus, when $\phi = 0$, the unique Subgame Perfect Nash Equilibrium of the static game is given by $l = 0$ and $k = \underline{k}^*_u$. Once $\phi > 0$, however, the buyer must make an initial transfer if a positive level of capacity is to be built. It is straightforward to check that in this case the optimal decision of the supplier is given by

$$k(l) = \min\left(\frac{l}{\phi c}, \underline{k}^*_u\right). \tag{5.2}$$

Therefore, when the initial payment is large enough so that $\underline{k}^*_u$ is feasible, the supplier chooses $\underline{k}^*_u$; otherwise he invests as much as allowed by his financing constraint.

The buyer's problem then reduces to choose an initial payment to maximize his expected profits anticipating the supplier's response, i.e.,

$$\max_{l \geq 0} \ \pi^B(l) = -l + (1-\beta^s)\,\mathbb{E}\left[R(k(l),\theta)\right], \tag{5.3}$$

where $k(l)$ is given by (5.2). The next proposition characterizes the equilibrium of the game under spot transactions.

Proposition 5.1 (Spot Transactions) *In the unique equilibrium under spot transactions:*

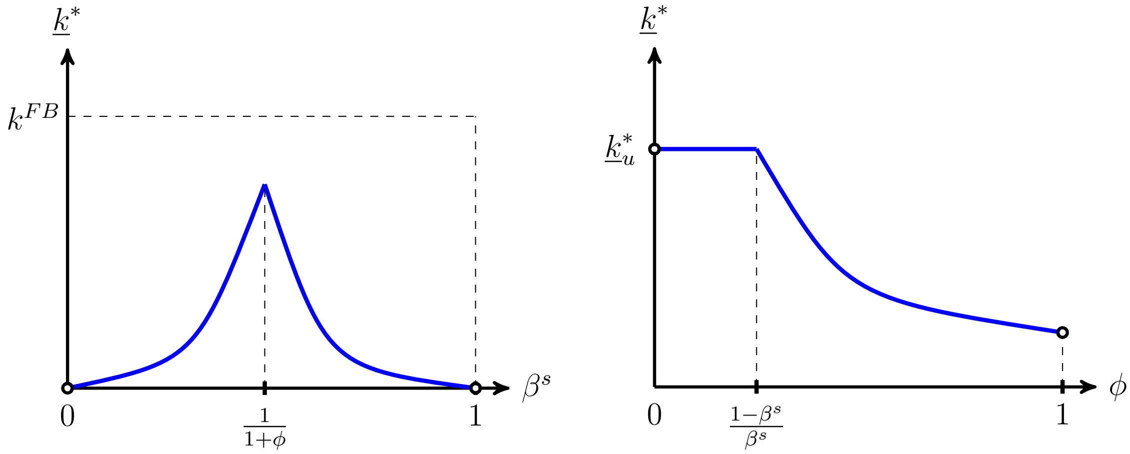
(i) The buyer makes an initial payment of $\phi c \underline{k}^*$, the supplier builds $\underline{k}^*$ units of capacity, and, conditional on market conditions being θ, the supplier receives a payment of $\beta^s R(\underline{k}^*, \theta)$ when the components are delivered, where $\underline{k}^* = \min(\underline{k}_u^*, \underline{k}_B^*)$, $\underline{k}_u^*$ is given by (5.1), and

$$\underline{k}_B^* := \arg\max_{k \geq 0} \ -\phi c k + (1-\beta^s)\,\mathbb{E}\left[R(k,\theta)\right]. \tag{5.4}$$

(ii) Investment in capacity is strictly lower than the efficient level, i.e., $\underline{k}^* < k^{FB}$.

The intuition behind Proposition 5.1 is simple. Remember that $\underline{k}_u^*$ denotes the investment level chosen by a financially unconstrained supplier. By paying a fraction ϕ of the initial cost, the buyer can then induce the supplier to choose any $k \leq \underline{k}_u^*$. On the other hand, note that $\underline{k}_B^*$ represents the buyer's optimal investment choice when the supplier can be forced to build any level of capacity provided that the financing constraint is satisfied. When both the supplier's bargaining power β^s and the financing friction (as measured by ϕ) are relatively large, $\underline{k}_B^* \leq \underline{k}_u^*$ and the buyer can implement $\underline{k}_B^*$, his preferred level of investment. Alternatively, when $\underline{k}_B^* > \underline{k}_u^*$, regardless of the initial payment, the supplier will never implement the buyer's optimal choice. The buyer knows this and will only transfer enough resources to build $\underline{k}_u^*$ units of capacity, the maximum level that the supplier is willing to implement.

Finally, part (ii) states that regardless of the specific values of β^s and ϕ, spot transactions always deliver a second-best outcome. When $\beta^s < 1$, the reason is related to the double-marginalization principle (see, e.g., Spengler (1950) and Rey and Tirole (1986)): the sequential structure of the game and the inability to write contracts imply that the supplier's cost structure becomes distorted once the transfer from the buyer is introduced. Intuitively, the supplier captures only a fraction β^s of the total revenue but from his perspective, he bears the entire investment cost (either from the loan or from the initial transfer, which he internalizes as his own). As a consequence, the supplier's choice of investment is smaller than k^{FB}. Alternatively, when $\beta^s = 1$, the supplier would like to choose k^{FB}. The buyer, however, gets no benefits from investment and is not willing to make any transfer to the



(A) Investment as a function of β^s (B) Investment as a function of ϕ

Figure 2: Comparative Statics of Investment under Spot Transactions

supplier. As a consequence, no investment is made.

The next result establishes some interesting comparative statics of the equilibrium under spot transactions. In particular, it shows how the level of capacity depends on the magnitude of the financing friction and the supplier's ex post bargaining power.

Proposition 5.2 (Comparative Statics - Investment) *In the unique equilibrium under spot transactions:*

(i) There is an inverted U-shaped relationship between investment and the supplier's ex post bargaining power. Specifically, k is strictly increasing in β^s for all $\beta^s \in [0, 1/(1+\phi))$ and strictly decreasing for all $\beta^s \in [1/(1+\phi), 1)$.

(ii) Investment in capacity is weakly decreasing in the magnitude of the financing friction. In particular, k is constant for all $\phi \in [0, \widehat{\phi}]$ and strictly decreasing in ϕ for all $\phi \in [\widehat{\phi}, 1)$, where $\widehat{\phi} = \min[1, (1-\beta^s)/\beta^s]$.

To understand the first part of the proposition, note that an increase in the supplier's ex post bargaining power has two opposite effects: it raises $\underline{k}_u^*$, but it also reduces $\underline{k}_B^*$. When β^s is relatively small so that $\underline{k}_B^* > \underline{k}_u^*$ (i.e., $\beta^s \leq 1/(1+\phi)$), investment is given by $\underline{k}_u^*$ (Proposition 5.1). An increase in β^S then raises investment. Alternatively, when β^s is relatively large so that $\underline{k}_B^* < \underline{k}_u^*$ (i.e., $\phi \leq (1-\beta^s)/\beta^s$), investment is given by $\underline{k}_B^*$. In this case, the increase in $\underline{k}_u^*$ is irrelevant and the decrease in the buyer's optimal choice lowers the equilibrium level of investment. The intuition behind the second part is similar and thus omitted. The proposition is illustrated in Figure 2.

Throughout the paper I assume that the revenue function satisfies the following requirement:

Assumption 1 *The elasticity of the slope of the revenue function is greater than minus one, i.e.,*

$$\epsilon(k,\theta) := \frac{kR_{kk}(k,\theta)}{R_k(k,\theta)} \geq -1,$$

for all $\theta \in \Theta$ and all $k \geq 0$.

This assumption is made for expositional purposes only. It guarantees that the buyer's optimal choice of investment is elastic with respect to ϕ, simplifying the relationship between the supplier's profits and the magnitude of the financing friction (see Proposition 5.3 below). Also note that the condition is valid for most commonly used production functions. For example, it holds when $R(k,\theta) = f(\theta)k^{\alpha}$, with $f' > 0$ and $\alpha \in (0,1)$.

The next proposition summarizes how the firms' profits depend on the magnitude of the financing friction.

Proposition 5.3 (Comparative Statics - Profits) *In the unique equilibrium under spot transactions:*

(i) Joint profits $\underline{\pi}^S + \underline{\pi}^B$ are weakly decreasing in ϕ. Specifically, they are constant in ϕ for all $\phi \in (0, \widehat{\phi})$ and strictly decreasing for all $\phi \in (\widehat{\phi}, 1]$, where $\widehat{\phi} = \min[1, (1-\beta^s)/\beta^s]$.

(ii) The buyer's profits $\underline{\pi}^B$ are strictly decreasing in ϕ.

(iii) The supplier's profits $\underline{\pi}^S$ are strictly increasing in ϕ for all $\phi \in (0, \widehat{\phi})$ and strictly decreasing for all $\phi \in (\widehat{\phi}, 1]$.

Part (i) holds because joint profits only depend on k and, by Proposition 5.2, investment is weakly decreasing in ϕ. Part (ii) reflects the fact that when ϕ rises, the buyer must pay a larger fraction of the investment cost. A less obvious fact (which will turn out to be important) is that $\underline{\pi}^S$ is non-monotone in ϕ. To develop some intuition, note that a higher ϕ has two opposite effects on the supplier's profits. On the one hand, if investment is held constant, the fraction of the cost covered by the supplier decreases, increasing his profits. On the other hand, as showed in Proposition (5.2), investment may decrease when ϕ rises, reducing the supplier's revenue. When $\phi \in (0, \widehat{\phi})$, k remains constant and only the first effect is present. The supplier's profits therefore strictly increase. Alternatively, when $\phi \in (\widehat{\phi}, 1]$, increasing ϕ decreases investment and both effects coexist. Which of the two dominates then depends on how large the change in k is. In the appendix I show that when Assumption 1 holds, the elasticity of investment with respect to ϕ is large enough for the second effect to prevail. The net impact of an increase in ϕ on the supplier's welfare is then negative.

For future reference, let $\underline{\pi}^B(\phi)$ and $\underline{\pi}^S(\phi)$ denote the respective equilibrium profits under spot transactions of the buyer and the supplier when the financing friction is ϕ.

6 Self-Enforcing Relational Contracts

When there is a good chance that the firms will engage in future transactions, the ongoing nature of the relationship can be used to provide incentives to enforce an implicit agreement and increase the efficiency of the relationship beyond that under spot transactions. Intuitively, because their current behavior will affect the nature of future interactions, firms have an incentive to conform to an implicit agreement. Because they are enforced by the value of the ongoing relationship rather than a court, these agreements are called *self-enforcing relational contracts.* In this section I offer a formal definition of such contracts.

Note that each period t is divided into 5 subperiods: t^1, in which the buyer offers the compensation scheme; t^2, in which the supplier decides on the buyer's offer; t^3, in which the supplier chooses the level of investment; t^4, in which the buyer makes an offer for the components ; and t^5, in which the supplier decides between the buyer's ex post offer and the bargaining outcome. Let $a_t = \{l_t, T_t, p_t, k_t, \theta_t, w_t, d_t\}$ denote the outcome of these decisions. The relevant histories of information up to date t are then given by $h(t^1) = (a_1, a_2, ..., a_{t-1})$, $h(t^2) = h(t^1) \cup \{l_t, T_t\}$, $h(t^3) = h(t^2) \cup \{p_t\}$, $h(t^4) = h(t^3) \cup \{k_t, \theta_t\}$, and $h(t^5) = h(t^4) \cup \{w_t\}$, with $h(1^1) = \emptyset$. Let $\mathcal{H}(t^1)$, $i \in \{1,2,3,4,5\}$, denote the set of all possible $h(t^i)$ histories. A *strategy of the buyer* (σ_B) is a sequence of functions $\{L_t, T_t, W_t\}_{t=1}^{\infty}$, where for each period t: i) $L_t : \mathcal{H}(t^1) \to \mathbb{R}_+$ and $T_t : \mathcal{H}(t^1) \cup \{l_t\} \times \Theta \to [\phi ck - l_t, \infty]$ specify the contract to be offered by the buyer; and ii) $W_t : \mathcal{H}(t^4) \to [\phi ck_t - l_t, \infty]$ specifies the ex post transfer offered to the supplier conditional upon the number of components delivered by the supplier and the realization of the demand parameter. Similarly, a *strategy of the supplier* is a sequence of functions $\{P_t, K_t, D_t,\}_{t=1}^{\infty}$, where for each period t: i) $P_t : \mathcal{H}(t^2) \to \{0,1\}$ specifies whether the supplier should accept the buyer's offer; ii) $K_t : \mathcal{H}(t^3) \to [0, l_t/\phi c]$ specifies the supplier's choice of investment; and iii) $D_t : \mathcal{H}(t^5) \to \{0,1\}$ specifies the supplier's decision regarding the ex post payment w_t. Let Σ_B and Σ_S denote the sets of strategies of the buyer and the seller, respectively, and let $\Sigma = \Sigma_B \times \Sigma_S$ denote the set of strategy profiles.

A *Subgame Perfect Equilibrium* (SPE in the following) is a profile of strategies $\sigma \in \Sigma$ that, conditional on any date t and any history $h(t^i)$, induces a Nash equilibrium for the repeated game i.e. $\pi^j\left[\sigma_j^*, \sigma_{-j}^* | h(t^i)\right] \geq \pi^j\left[\sigma_j, \sigma_{-j}^* | h(t^i)\right]$, $j \in \{B, S\}$, for all $t \geq 1$, $i \in \{1,2,3,4,5\}$, $h(t^i) \in \mathcal{H}(t^i)$, and $\sigma_j \in \Sigma_j$, where $\pi^j[\sigma | h(t^i)]$ denotes firm j's expected discounted profits from date t^i on given the strategy profile σ.

Now let $h_e^t = (\theta_1, \theta_2, ..., \theta_{t-1})$ denote the *exogenous* history of demand realizations up to period t and let $\mathcal{H}_e^t$ denote the set of such histories. As usual, let the initial history h_e^0 be equal to the empty set. A relational contract is a complete plan for the relationship.

Definition 1 *A **relational contract** is a feasible contingent plan*

$$\alpha := \left\{L_t\left(h_e^t\right), k_t\left(h_e^t\right), T_t\left(h_e^{t+1}\right)\right\}_{t=1}^{\infty},$$

where for each date t and every history $h_e^t \in \mathcal{H}_e^t$: (i) $l_t\left(h_e^t\right)$ specifies the initial payment to the supplier; (ii) $k_t\left(h_e^t\right)$ specifies the level of capacity to be built by the supplier; and (iii) $T_t\left(h_e^{t+1}\right)$ specifies the ex post transfer to the supplier conditional upon the realization of the demand parameter at time t.

Given a relational contract α and any *t-period* history h_e^t, the buyer's expected continuation profits at the beginning of the period are given by

$$\pi_t^B\left(\alpha|h_e^t\right) = (1-\delta)\mathbb{E}\left[\sum_{\tau=t}^{\infty}\delta^{\tau-t}\left\{-l_\tau\left(h_e^\tau\right) + R\left[k_\tau\left(h_e^\tau\right), \theta_\tau\right] - T_\tau\left(h_e^\tau, \theta_\tau\right)\right\}\right],$$

where the expectation is taken with respect to the probability distribution induced by $(f_\theta)_{\theta\in\Theta}$ over the set of histories and I multiply by $(1-\delta)$ to express payoffs as per-period averages. Similarly, the supplier's expected continuation profits are given by

$$\pi_t^S\left(\alpha|h_e^t\right) = (1-\delta)\mathbb{E}\left[\sum_{\tau=t}^{\infty}\delta^{\tau-t}\left\{l_\tau\left(h_e^\tau\right) - ck_\tau\left(h_e^\tau\right) + T_\tau\left(h_e^\tau, \theta_\tau\right)\right\}\right].$$

Definition 2 *A relational contract is **self-enforcing** if it coincides with the outcome path generated by a Subgame Perfect Equilibrium of the repeated game.*

Following the original idea of Abreu (1988), the next proposition provides necessary and sufficient conditions for a self-enforcing relational contract. More precisely, it shows that any self-enforcing relational contract can be supported by the threat that any deviation from equilibrium actions will trigger a permanent reversion to the equilibrium that yields the lowest payoffs to the deviator (which, in the current set up, coincides with spot transactions for both parties). Intuitively, any strategy profile of the repeated game can be viewed as a rule specifying an outcome (or equilibrium) path and punishments for deviations from the outcome path or from previously specified punishments. If the strategy profile is an equilibrium, the punishments are such that no firm has incentives to deviate. This, however, remains true if the original punishment is replaced by the SPE that minimizes the deviator's payoffs. After all, the punishment for a deviation becomes stronger.

Proposition 6.1 *A relational contract α is self-enforcing if and only if for all periods $t \geq 1$ and all histories $h_e^t \in \mathcal{H}_e^t$:*

$$\pi_t^B\left(\alpha|h_e^t\right) \geq \underline{\pi}^B, \tag{6.1}$$

$$\pi_t^S\left(\alpha|h_e^t\right) \geq (1-\delta)\,\pi_{spot}^S\left[l_t\left(h_e^t\right)\right] + \delta\underline{\pi}^S, \tag{6.2}$$

$$\delta\left[\pi_t^B\left(\alpha|h_e^t,\theta\right) - \underline{\pi}^B\right] \geq (1-\delta)\left\{T_t\left(h_e^t,\theta\right) - \beta^s R\left[k_t\left(h_e^t\right),\theta\right]\right\},\ \forall\theta\in\Theta, \tag{6.3}$$

$$\delta\left[\pi^S\left(\alpha|h_e^t,\theta\right) - \underline{\pi}^S\right] \geq (1-\delta)\left\{\beta^s R\left[k_t\left(h_e^t\right),\theta\right] - T_t\left(h_e^t,\theta\right)\right\},\ \forall\theta\in\Theta, \tag{6.4}$$

$$l_t\left(h_e^t\right) \geq \phi c k_t\left(h_e^t\right) \tag{6.5}$$

$$T_t\left(h_e^t,\theta\right) \geq \phi c k_t\left(h_e^t\right) - l_t\left(h_e^t\right),\ \forall\theta\in\Theta, \tag{6.6}$$

where

$$\pi_{spot}^S\left[l_t\left(h_e^t\right)\right] := \arg\max_{k\geq 0}\ \pi_{spot}^S\left[l_t\left(h_e^t\right),k\right].$$

Equations (6.1)-(6.4) capture the inefficiencies introduced by the lack of commitment. In particular, equation (6.1) ensures that the buyer has the incentives to make the initial payment specified by the relational contract. Equation (6.2), on the other hand, guarantees that the supplier, after receiving the initial payment, is willing to build the recommended level of capacity. Finally, equations (6.3) and (6.4) ensure that, once capacity is built and demand conditions are realized, both firms have the incentives to honor the transfers specified in the contract instead of exploiting their bargaining power. Note that, as mentioned earlier, any deviation is punished with perpetual reversion to spot transactions. Equation (6.5) is the financing constraint. Equation (6.6), the supplier's liquidity constraint, guarantees that ex post transfers respect the fact that the supplier has no resources other than those left after investing in capacity.

There are of course many possible self-enforcing contracts, so I focus on those which are Pareto efficient. A self-enforcing relational contract ψ is *efficient* if it is not Pareto dominated by any other self-enforcing relational contract, i.e., if there is no other self-enforcing contract ψ' such that $\pi^B(\psi') \geq \pi^B(\psi)$ and $\pi^S(\psi') \geq \pi^S(\psi)$, with at least one strict inequality. Note that a efficient self-enforcing relational contract that guarantees the supplier discounted expected profits of π^S solves the program

$$\max_{\psi\in\Psi}\ \pi^B(\psi|h_e^1) \text{ subject to } \pi^S(\psi|h_e^1) \geq \pi^S, \tag{6.7}$$

where Ψ stands for the set of self-enforcing relational contracts. In the remainder of the paper I characterize Ψ^*, the set of efficient self-enforcing contracts.

6.1 Recursive Formulation

To solve for an efficient self-enforcing relational contract, one has to find the equilibrium that maximizes the utility of one firm subject to delivering at least a certain value to the other. Unfortunately, since both firms will condition their actions on the entire history of play, this task can potentially be very complicated. It is well known, however, that in the setting under study, every relational contract has a recursive representation, with continuation profits to one of the firms as state variable.[7] Essentially, continuation profits summarize the entire history of public information, making it possible to solve the problem using standard dynamic programming techniques.

Let $Q(\pi^S)$ denote the maximum equilibrium profits to the buyer when the supplier's current promised continuation profits are π^S. The next proposition formalizes the previous discussion by showing that efficient relational contracts solve a Bellman equation in which the state variable is π^S and the choice variables are the initial payment l, the level of capacity k to be built by the supplier, and, given that no firm has deviated, current transfers $T(\theta)$ and future continuation profits to the buyer $\pi^S(\theta)$ conditional upon the current's period realization of the demand parameter θ.[8]

Proposition 6.2 *An efficient self-enforcing relational contract that guarantees the supplier discounted expected profits of π^S solves the following functional equation:*

$$Q\left(\pi^S\right) = \sup_{k,l,T(\theta),\pi^S(\theta)} (1-\delta)\left\{-l + \mathbb{E}\left[R(k,\theta) - T(\theta)\right]\right\} + \delta\mathbb{E}\left\{Q\left[\pi^S(\theta)\right]\right\} \tag{P1}$$

[7]See, e.g., Green (1987), Spear and Srivastava (1987), Thomas and Worrall (1988), Abreu, Pearce, and Stacchetti (1990), Phelan and Townsend (1991), and Thomas and Worrall (1994). A detailed discussion of this technique is beyond the scope of the paper, but the general idea is to consider an equilibrium as specifying actions for every player in period one and continuation strategies in the future. Since the continuation strategies are common knowledge, they constitute an equilibrium. This implies that it is possible to "factorize" any equilibrium into a pair of first period actions and continuation utilities and suggests the use of promised utilities as state variables. For a more general discussion see Abreu, Pearce, and Stacchetti (1990).

[8]For notational ease, I frequently suppress dependence of the contract on π^S

subject to

$$(1-\delta)\left\{l - ck + \mathbb{E}\left[T(\theta)\right]\right\} + \delta\mathbb{E}\left[\pi^S(\theta)\right] \geq \pi^S, \tag{P-K}$$

$$(1-\delta)\left\{l - ck + \mathbb{E}\left[T(\theta)\right]\right\} + \delta\mathbb{E}\left[\pi^S(\theta)\right] \geq (1-\delta)\,\pi^S_{spot}(l) + \delta\underline{\pi}^S, \tag{NR-S-K}$$

$$(1-\delta)\left[\beta^s R(k,\theta) - T(\theta)\right] + \delta Q\left[\pi^S(\theta)\right] \geq \delta\underline{\pi}^B,\ \forall\theta\in\Theta, \tag{NR-B$_\theta$}$$

$$(1-\delta)\left[T(\theta) - \beta^s R(k,\theta)\right] + \delta\pi^S(\theta) \geq \delta\underline{\pi}^S,\ \forall\theta\in\Theta, \tag{NR-S$_\theta$}$$

$$l \geq \phi ck, \tag{F-C}$$

$$T(\theta) \geq \phi ck - l,\ \forall\theta\in\Theta, \tag{L-C$_\theta$}$$

$$\pi^S(\theta) \in [\underline{\pi}^S, \pi^S_{\max}],\ \forall\theta\in\Theta. \tag{C-C$_\theta$}$$

Equation (P-K) is the *promise-keeping condition,* which requires the contract to deliver (at least) the promised level of profits to the supplier. Note that $l - ck$ is the actual amount paid by the supplier for the current investment. This amount need not equal $(1-\phi)ck$ as the buyer is free to contribute a larger fraction of the investment cost than is required by the supplier's financing constraint. Equations (NR-S-K), (NR-B$_\theta$) and (NR-S$_\theta$), the *non-reneging constraints*, are the recursive versions of (6.2), (6.3) and (6.4), respectively. As mentioned earlier, they ensure that the supplier is willing to build the recommended level of capacity and that, once capacity is built and demand conditions are realized, both firms are willing to honor the ex post transfers specified in the contract instead of exploiting their bargaining power. Equations (F-C) and (L-C$_\theta$) are the *financing and liquidity constraints.* Finally, equation (C-C$_\theta$), the *credibility constraint*, guarantees that the supplier's continuation profits are credible in the sense of being equilibrium profits. Observe that the minimum equilibrium payoff to the supplier is $\underline{\pi}^S$. This hold since, as noted earlier, perpetual reversion to spot transactions is the worst equilibrium outcome for both firms. Also note that the set of equilibrium payoffs to the supplier is both compact and convex. This follows from standard arguments. For future reference, a contract $\left\{k, l, T(\theta), \pi^S(\theta)\right\}$ is said to be *valid* if it satisfies all previous constraint but promise-keeping.

Two features of Problem (P1) are worth noting. First, it is written from the perspective of the buyer. This is without loss of generality since the whole set of efficient contracts can be traced out by varying π^S. Second, the buyer's future continuation profits are given by the function Q. That is, continuation payoffs to the firms lie in the Pareto frontier. The reason is that efficient contracts cannot be Pareto dominated after any positive probability history. Intuitively, replacing any part of a contract that is dominated by its Pareto dominant counterpart would increase the payoffs to both firms while (weakly) relaxing all constraints. Thus, contradicting the Pareto efficiency of the original contract.

Lastly, note that if the promise-keeping condition is slack, the solution to Problem (P1) is independent of the value of π^S. The next result shows that this may happen in a neighborhood of $\underline{\pi}^S$.

Lemma 6.3 *There exists a $\pi^S_{\min} \in \left[\underline{\pi}^S, \pi^S_{\max}\right)$ such that the promise-keeping constraint is slack for any $\pi^S < \pi^S_{\min}$ and is binding for any $\pi^S \geq \pi^S_{\min}$. Moreover, $Q'\left(\pi^S\right) = 0$ for all $\pi^S \in \left(\underline{\pi}^S, \pi^S_{\min}\right)$.*

Intuitively, due to the liquidity constraints, the boundary of the set of equilibrium payoffs can be upward sloping in the supplier's continuation value when the latter is close to $\underline{\pi}^S$. Consequently, the buyer can benefit from giving the supplier a higher continuation value. This also implies that, in any efficient relational contract, the supplier earns profits above the threshold $\pi^S_{\min}$, which may be larger than $\underline{\pi}^S$.

7 A Benchmark: Unconstrained Supplier

To establish a benchmark against which to evaluate the impact of the supplier's financing constraint on the structure of efficient contracts, consider first the case in which the supplier can fully cover the cost of investment, i.e., $\phi = 0$.[9] In order to do so, assume for the moment that, if required, the supplier can also transfer any amount of resources to the buyer—both before and after investment takes place. Later I show that this is never the case, but making such assumption at this point allows me to greatly simplify the problem at hand by following Levin (2003) and, without loss of generality, restricting attention to stationary contracts, i.e., contracts in which, on the equilibrium path, firms always agree on the same compensation scheme and the supplier always chooses the same level of investment.[10] Intuitively, when liquidity constraints are not a concern, any variation in continuation utilities required by a non-stationary contract can be replaced with variations in current payments without affecting incentives.

I begin by finding the relational contract that maximizes the buyer's profits and then use it to characterize efficient contracts along the whole Pareto frontier. To find the buyer's optimal relational contract, first note that the supplier's profits must be at its lowest level (i.e., $\pi^S[k, l, T(\theta)] = \underline{\pi}^S(0)$), otherwise it would be possible to increase π^S by decreasing the fixed payment at the beginning of the first period. Constraint (NR-S-K) then reduces

[9] The following analysis is closely related to work by Taylor and Plambeck (2007b), who characterize the buyer's optimal contract in a similar model under the assumption that *both* firms have access to a large amount of resources.

[10] Although Levin (2003) assumes that the buyer has all the bargaining power, it is straightforward to generalize his result to the current set up.

to $l \leq 0$ and the problem can be written as

$$\max_{k,l,T(\theta)} \quad -l + \mathbb{E}\left[R\left(k,\theta\right) - T\left(\theta\right)\right]$$

subject to

$$l - ck + \mathbb{E}\left[T\left(\theta\right)\right] = \underline{\pi}^{S}\left(0\right), \tag{7.1}$$

$$l \leq 0, \tag{7.2}$$

$$\begin{aligned} & -l + \mathbb{E}\left[R\left(k,\theta\right) - T\left(\theta\right)\right] - \underline{\pi}^{B}\left(0\right) \\ \geq & \frac{1-\delta}{\delta}\left[T\left(\theta\right) - \beta^{s} R\left(k,\theta\right)\right], \text{ for all } \theta \in \Theta, \end{aligned} \tag{7.3}$$

$$T\left(\theta\right) \geq \beta^{s} R\left(k,\theta\right), \text{ for all } \theta \in \Theta, \tag{7.4}$$

where recall that $\underline{\pi}^{B}(0)$ and $\underline{\pi}^{S}(0)$ denote, respectively, the buyer's and the supplier's expected profits under spot transactions when the supplier is not financially constrained.

Observe that if a non-trivial solution exists (i.e., $k > \underline{k}_u^*$), equation 7.4 must hold with strict inequality for at least one value of θ. Otherwise the supplier's participation constraint would require a positive fixed payment, contradicting constraint (NR-S-K). This in turn implies that it is without loss of generality to set the initial payment equal to zero. To see why, suppose that $l < 0$ and let θ' be a realization of market conditions for which $T(\theta') > \beta^s R(k,\theta')$ (which exits by the previous argument). The result then holds because it is always possible to increase l and decrease $T(\theta')$ so as to leave both firms with the same expected profits without violating any constraint. Finally, note that because $\pi^J[k,l,T(\theta)] = \pi^B[k,l,T(\theta)] + \pi^S[k,l,T(\theta)]$ and $\pi^S[k,l,T(\theta)] = \underline{\pi}^S(0)$, maximizing the buyer's profits is equivalent to maximizing total profits.

To further simplify the problem, note that it is also without loss of generality to let ex post transfers take the form of the supplier's disagreement payoff plus a constant bonus (i.e., $T(\theta) = \beta^s R(k,\theta) + A$), where the constant term can be pinned down by substituting $T(\theta)$ into the supplier's participation constraint:

$$T\left(\theta'\right) = \beta^{s} R\left(k,\theta'\right) \underbrace{-\beta^{s}\mathbb{E}\left[R\left(k,\theta\right)\right] + \underline{\pi}^{S}\left(0\right) + ck}_{A}, \text{ for all } \theta' \in \Theta. \tag{7.5}$$

To understand why, suppose that there are two realizations of market conditions, say θ' and θ'', such that $T(\theta') - \beta^s R(k,\theta') > T(\theta'') - \beta^s R(k,\theta'')$. By (7.4), the left-hand side of the previous inequality is strictly larger than zero and, since the left-hand side of (7.3) is independent of the value of θ, the inequality in (7.3) is strict for θ''. Now consider an alternative contract with a smaller $T(\theta')$ and a larger $T(\theta')$ so that the expected value of the transfers remains unchanged. Easy calculations then show that the new contract does not affect the firms' profits and satisfies all the constraints, proving the result.

To summarize, the problem of finding the buyer's optimal relational contract reduces to maximizing joint surplus (i.e., $-ck+\mathbb{E}[R(k,\theta)]$) subject to (7.3) and (7.5). The next proposition characterizes the solution to this problem. The first part follows from the previous discussion and the fact that joint profits are maximized at k^{FB}. The second part is proved in the Appendix.

Proposition 7.1 (Buyer's Optimal Relational Contract) *Assume that the supplier is not financially constrained (i.e., $\phi = 0$) and let $\overline{k}$ denote the maximum value of k satisfying*

$$\underbrace{-ck+\mathbb{E}[R(k,\theta)]}_{\pi^J(k)} - \underline{\pi}^B(0) - \underline{\pi}^S(0) \geq \frac{1-\delta}{\delta}\left[-\beta^s \mathbb{E}[R(k,\theta)] + \underline{\pi}^S(0) + ck\right]. \tag{7.6}$$

(i) A stationary relational contract in which the initial payment is zero, i.e., $l_u^* = 0$, the supplier builds $k_u^* = \min\left(\overline{k}, k^{FB}\right)$ units of capacity, and the buyer pays the supplier the voluntary ex post transfer specified in equation (7.5) maximizes the buyer's profits over the set of all self-enforcing relational contracts.

(ii) Any self-enforcing relational contract that achieves the buyer's maximum equilibrium profits requires, after any period and any history, a non-positive initial payment.

The structure of the contract is very simple. The buyer offers no initial payment and instead guarantees a fixed transfer plus a share β^s of the total revenue. Many existing papers have identified situations in which profit sharing contracts are optimal (for some early wok, see, e.g., Cheung (1969) and Stiglitz (1974)). Most of them, however, explain the optimality of this type of contracts using models of risk sharing or moral hazard. Proposition 7.1 suggests an alternative explanation. Namely, that share contracts can arise as an optimal response to contract incompleteness. Unfortunately, Proposition 7.1 does not establish that this is the only way to implement the buyer's optimal relational contract. Intuitively, since both players are risk neutral and discount future profits at the same rate, there is some degree of indeterminacy on the optimal repayment policy. Nevertheless, regardless of the way transfers are arranged, no initial payment takes place under the buyer's profits maximizing contract. This result will be important to understanding how financial frictions affect the structure of efficient relational contracts.

The next result characterizes the whole Pareto frontier when the supplier does not face financing constraints.

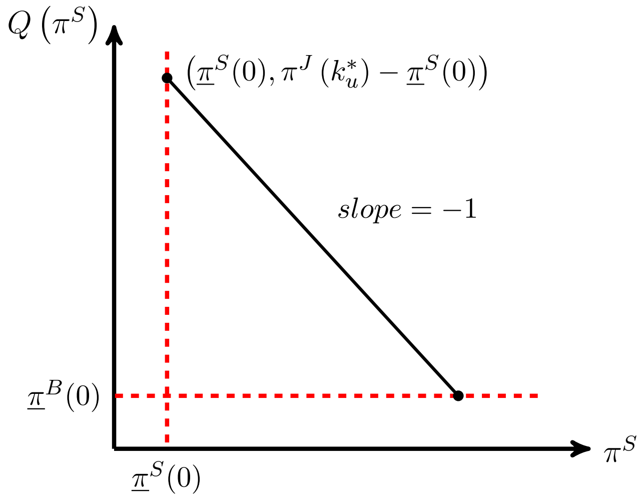


Figure 3: Pareto Frontier w/o Financing Frictions: $\phi = 0$

Theorem 7.2 *Let $\pi^J\left(k_u^*,\right)$ denote the expected joint profits under the buyer's optimal relational contract, i.e., $\pi^J\left(k_u^*,\right) := -ck_u^* + \mathbb{E}\left[R\left(k_u^*, \theta\right)\right].$*

(i) There exist efficient self-enforcing relational contracts generating any profile of profits $\left(\pi^S, \pi^B\right)$ satisfying $\pi^S \geq \underline{\pi}^S(0)$, $\pi^B \geq \underline{\pi}^B(0)$, and $\pi^S + \pi^B \leq \pi^J\left(k_u^*,\right).$

(ii) For any efficient self-enforcing contract, there exists a stationary contract that achieves the same profits for both firms.

The first part of Theorem 7.2 states that if the supplier can raise enough resources to fully pay the initial costs of investment ($\phi = 0$), then there is no conflict between maximizing the surplus generated by the relationship and the way such surplus is divided between the firms. Similar to Theorem 1 in Levin (2003), the reasoning behind part (i) is that when the supplier is not financially constrained, transfers can always be used to redistribute surplus without affecting the firms' incentives to honor the agreement. It is worth stressing, however, that the result is not a special case of Levin (2003). The latter assumes that both agents are liquidity unconstrained. Consequently, the *initial transfers in the first period of the contract* can always be adjusted to achieve any division of surplus without affecting the incentives *after such payment is made*. In the current set up, this may not be feasible. In particular, it may not be possible to decrease the initial payment because it is already at its lowest level (i.e., zero) or because doing so would violate the supplier's liquidity constraints. Levin's argument may thus fail when trying to increase the buyer's share of the total surplus. Theorem 7.2 shows that if that is the case, it is nevertheless possible to accomplish the same result by modifying the division of ex post gains from investment instead.

The second part of the Theorem establishes that to achieve any efficient profile of profits, firms can employ contracts that, on the equilibrium path, offer the same conditions in each period. Intuitively, incentives are provided by a combination of voluntary transfers and future terms of the contract (continuation profits). But part (i) implies that the cost of delaying or advancing compensation—which is given by the slope of Q—is the same as the cost of current compensation (i.e., -1). Thus, instead of modifying the future terms of trade, the parties can "settle up" each period using discretionary payments and then proceed to the same optimal agreement at the next date. A similar result was also proved by Levin (2003) but, as mentioned earlier, assuming no liquidity constraints. My result then complement Levin's by showing that—under certain conditions—one can still restrict attention to stationary contracts even when the ability of one of the firm to make transfers is limited by liquidity considerations.

8 Characterization of Efficient Contracts

The second part of Theorem (7.1) shows that when the supplier can fully pay the cost of investment, the buyer's most preferred relational contract involves delaying all payments until the components are delivered. When the supplier is financially constrained, however, such agreement would rule out the possibility of undertaking any investment, resulting in zero profits for both firms. Efficient transfers and investment policies when financial markets are perfect therefore will not remain efficient once the buyer needs to finance part of the initial cost of investment. In this section, I analyze how the presence of financing frictions affects the structure of efficient relational contracts.

8.1 Preliminary Results

Before turning to the characterization of the set of efficient self-enforcing contracts, I begin with some preliminary results that will be useful through the paper. The next lemma discusses some key properties of the value function Q.

Lemma 8.1 *The value function $Q(\pi^S)$ is (i) concave, (ii) strictly decreasing on $\left[\pi^S_{\min}, \pi^S_{\max}\right]$, and (iii) has slope larger than -1.*[11]

The concavity of Q implies that the cost of providing an additional unit of value to the supplier is increasing in π^S. Intuitively, while a higher π^S requires more transfer to the supplier, it also relaxes the supplier's liquidity situation, allowing the buyer to implement a more

[11] The value function Q may not be differentiable, in which case the last part of the proposition should be interpreted as $\min \partial Q(\pi^S) \geq -1$, where $\partial Q(\pi^S)$ stands for the superdiferential of the value function Q at π^S. Observe that $\partial Q(\pi^S)$ is never empty since Q is concave.

favorable agreement. As π^S moves away from $\underline{\pi}^S$, however, the first effect remains the same but the second effect decreases. The second part of the lemma shows that increasing the supplier's profits is always costly to the buyer. This holds because Pareto efficient self-enforcing contract are sequentially Pareto efficient, i.e., all continuation profits required to generate an equilibrium in the Pareto frontier belong themselves to the Pareto frontier. Intuitively, since all information is common knowledge, substituting Pareto dominated continuation profits by their Pareto efficient counterparts would not violate any of the constraints and would induces a Pareto improvement. Thus, contradicting the efficiency of the original contract. Finally, part (iii) shows that the cost of a one-unit increase in π^S never exceeds one. This holds because the buyer has "deep pockets". Intuitively, by increasing the initial or the ex post payments, the buyer can always transfers profits to the supplier at a rate of one-to-one without violating any of the constraints. The buyer may, however, to do better by changing the level of capacity or the continuation profits.

The last part of the Lemma also implies that the total value of the relationship, i.e. $Q(\pi^S)+\pi^S$, is increasing in the supplier's continuation profits. The reason is that a higher π^S relaxes the supplier's financing and liquidity constraints, reducing distortions on investment. Note that by part (iii), a higher π^S also implies lower profits to the buyer. One could then wonder whether the latter's incentives to renege on the agreement increase. And if they do, why this does not negatively affect investment. Interestingly, because the buyer is not liquidity constrained, it is always possible to use a combination of lower ex post transfers and a higher initial payment to increase the supplier's profits without affecting the former's non-reneging constraints.[12] This property will turn out to be important later on to understand the dynamics of investment across efficient relational contracts.

8.2 Socially Optimal Contracts

A self-enforcing contract is *socially optimal* if it maximizes the social surplus created by the relationship (discounted expected joint profits) across all possible self-enforcing contracts. Let Ψ^o denote the set of such contracts, i.e.,

$$\Psi^o := \arg\max_{\psi\in\Psi} \ \pi^B(\psi) + \pi^S(\psi).$$ [13]

Note that all socially optimal contracts are efficient. The converse, however, may not be true. Remember from Proposition 8.1 that the slope of Q is decreasing in π^S (since Q is

[12] It is possible to prove that the increment in l must be larger than the decrease in the ex post transfers so that the supplier's liquidity constraint is not violated.

[13] That Ψ^o is well defined follows from the compacted of the set of equilibrium payoffs.

concave) and always weakly larger than -1. Thus, as long as the slope of Q is strictly larger than -1, the value of the relationship continues to grow with π^S and efficient contracts are not socially optimal.

Definition 3 *Let $\widehat{\pi}^S$ denote the minimum value of the supplier's discounted expected profits above which efficient contracts are also socially optimal, i.e.,*

$$\widehat{\pi}^S = \begin{cases} \inf\left\{\pi^S \middle| \; Q'\left(\pi^S\right) = -1\right\}, & \text{if } Q'_-\left(\pi^S_{\max}\right) = -1 \\ \pi^S_{\max}, & \text{otherwise,} \end{cases}$$

where $Q'_-\left(\pi^S\right)$ stands for the left derivative of Q at π^S.

The next result will greatly simplify the task of characterizing socially optimal self-enforcing contracts.

Proposition 8.2 (Socially Optimal Contracts)

(i) If $\pi^S \geq \widehat{\pi}^S$, then $\pi^S(\theta) \in \left[\widehat{\pi}^S, \pi^S_{\max}\right]$ for all $\theta \in \Theta$.

(ii) For any socially optimal contract, there is a stationary contract that achieves the same profits for both firms.

The first part of the Proposition establishes that socially optimal contracts are in fact *sequentially optimal*. Intuitively, moving away from socially optimal contracts destroys joint surplus and offers no benefit in terms of relaxing any of the constraints. A sketch of the proof is as follows. Suppose that there is a socially optimal contract in which the supplier's continuation profits from period 2 onward, conditional on the demand parameter being θ, is $\pi^S_2(\theta) < \widehat{\pi}^S$. For simplicity, also assume that in such contract, the buyer's profits are strictly above his outside option. Note that the ex post transfer must be at its lowest level. This holds because, for any $\pi^S < \widehat{\pi}^S$, the slope of Q is strictly less than -1. Therefore, if $T(\theta)$ were not minimal, it would be possible to increase π^S and decrease $T(\theta)$ in such a way that both the supplier' profits and his non-reneging constraints remain unchanged but the buyer's profits strictly increase. If $T(\theta)$ is minimal, however, the buyer's ex post non-reneging constraint must be strictly slack. It should be clear then that just increasing $\pi^S_2(\theta)$ would induce a valid contract in which the cost of a one-unit increase in the supplier's profits is strictly less than one, increasing social surplus and contradicting the fact that the original contract is socially optimal.

The second part of the Proposition shows that any socially optimal contract can be implemented by a sequence of contracts that, on the equilibrium path, offers the same conditions in each period. As mentioned earlier, when firms can renege on their promises, incentives are provided through a combination of voluntary transfers and future terms of the contract (continuation profits). When the supplier's profits are large enough and the is socially optimal ($\pi^S \in \left[\widehat{\pi}^S, \pi^S_{\max}\right]$), part (i) ensures that the cost of delaying or advancing compensation—which is given by the slope of Q—is the same as the cost of current compensation (i.e., -1). The only potential catch is that the supplier's liquidity constraint may prevent the use of transfers as a substitute for lower continuation profits reach π^S. Because optimal contracts occur when the supplier's profits are relatively high, however, the latter must be receiving strictly positive transfers. Firms can then "settle up" each period using current transfers and then proceed to the same optimal agreement at the next date.

Proposition 8.2 implies that socially optimal contracts solve the following program:

$$\max_{(k,l,T(\theta))} -ck + \mathbb{E}\left[R\left(k,\theta\right)\right] \tag{P-SO}$$

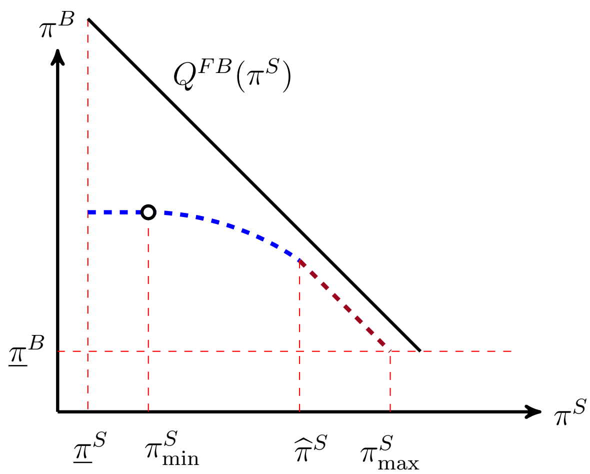
subject to

$$\begin{aligned}
&-l + \mathbb{E}\left[R\left(k,\theta\right) - T\left(\theta\right)\right] \geq \underline{\pi}^B, && \text{(PC-B)}\\
&l - ck + \mathbb{E}\left[T\left(\theta\right)\right] \geq \left(1-\delta\right)\pi^S_{spot}\left(l\right) + \delta\underline{\pi}^S, && \text{(NR-S-K)}\\
&\left(1-\delta\right)\left[\beta^s R\left(k,\theta\right) - T\left(\theta\right)\right] + \delta\left\{-l + \mathbb{E}\left[R\left(k,\theta\right) - T\left(\theta\right)\right]\right\} \geq \delta\underline{\pi}^B,\ \forall\theta\in\Theta, && \text{(NR-B-}\theta\text{)}\\
&\left(1-\delta\right)\left[T\left(\theta\right) - \beta^s R\left(k,\theta\right)\right] + \delta\left\{l - ck + \mathbb{E}\left[T\left(\theta\right)\right]\right\} \geq \delta\underline{\pi}^S,\ \forall\theta\in\Theta, && \text{(NR-S-}\theta\text{)}\\
&l - \phi ck \geq 0, && \text{(F-C)}\\
&T\left(\theta\right) + l - \phi ck \geq 0,\ \forall\theta\in\Theta, && \text{(C-C)}
\end{aligned}$$

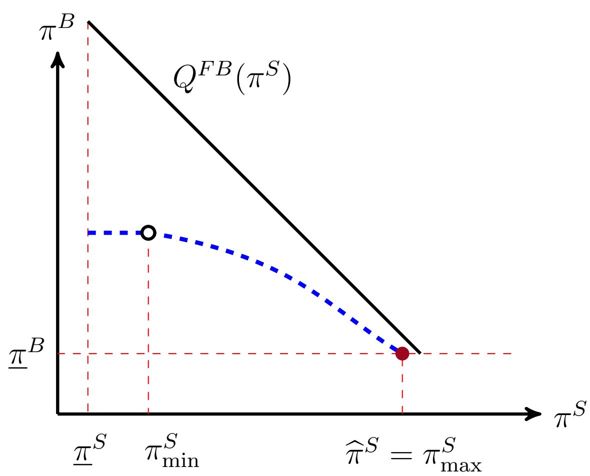
where constraint (PC-B), the buyer's participation constraint, has been added to ensure that the buyer's profits are never below his payoffs under spot transactions. The rest of the section is devoted to solving this problem.

Note that when the supplier's profits are maximal, the buyer's profits must be at its lowest level (i.e., $\pi^B\left(\pi^S_{\max}\right) = \underline{\pi}^B$). If this were not true, raising the fixed payment at the beginning of period 1 would increase π^S without violating any constraint, contradicting the definition of $\pi^S_{\max}$. It is clear then that there is always a socially optimal contract in which the buyer just breaks even, i.e., $\pi^B = \underline{\pi}^B$. Depending on the parameters of the model, such agreement can be the unique socially optimal contract (i.e., $\widehat{\pi}^S = \pi^S_{\max}$) or there may be others in which the buyer receives profits strictly above his outside option (i.e., $\widehat{\pi}^S < \pi^S_{\max}$). Figure 4 illustrates how the Pareto frontier looks in each case.

Definition 4 *If $\widehat{\pi}^S < \pi^S_{\max}$, I will say that socially optimal contracts are interior . If $\widehat{\pi}^S = \pi^S_{\max}$, on the other hand, I will say that there is a unique corner socially optimal contract.*



(A) Interior Socially Optimal Contracts



(B) Corner Socially Optimal Contract

Figure 4: The Value Function Q

Because social surplus only depends on the level of investment, an immediate implication of Proposition 8.2 is that all socially optimal contracts must feature the same time-invariant k. A natural questions is then how such level relates to k^{FB}. As the next Proposition shows, the answer turns out to depend critically on whether socially optimal contracts are interior.

Proposition 8.3 *Let k^o denote the level of investment under any socially optimal contract.*

(i) There is no over-investment, i.e., $k^o \leq k^{FB}$.

(ii) If socially optimal contracts are interior (i.e., $\widehat{\pi}^S < \pi^S_{\max}$), then k^o coincides with the investment level without financing frictions (i.e., $\phi = 0$).

(iii) If there is a unique corner socially optimal contract (i.e., $\widehat{\pi}^S = \pi^S_{\max}$), then k^o is

implicitly given by

$$\pi^J(k^o) - \underline{\pi}^J(\phi) = \frac{1-\delta}{\delta}\left[\max\left\{\pi^S_{spot}(\phi c k^o), \omega(k^o)\right\} - \pi^S(k^o)\right],$$

where

$$\omega(k^o) := \frac{-(1-\phi)ck^o - \left[1 - \Pr(\theta > \theta^*)\right]\pi^S(\phi)}{\Pr(\theta > \theta^*)} + \beta^s\left[R(k^o, \theta)\,|\theta > \theta^*\right],$$

and θ^* *is the largest* $\theta \in \Theta$ *such that* $\pi^J(k^o) - \underline{\pi}^J(\phi) > \frac{1-\delta}{\delta}\beta^s R(k,\theta)$.

The first part of the Proposition shows that socially optimal contracts never feature over-investment. Intuitively, if $k^0 > k^{FB}$, then it would be possible to decrease investment and use transfers to divide the extra surplus in such that way that both firms have the incentives to honor the new agreement. As a consequence, total surplus could be raised, contradicting the optimality of the original contract. Because the supplier has no wealth and his access to credit is limited, the previous argument requires the supplier to be receiving *rents from the relationship*. As I show in the next section, this will not always be true. However, it is true for socially optimal contracts as they occur when the supplier's profits are relatively large. Remember from Proposition 4.1 that without contracting frictions, investment is always at its first-best level. An empirical implication of part (i) is then that firms must be bigger in regions with better contract enforcement. This prediction is consistent with empirical evidence. For instance, using data from a census of Mexican firms, Laeven and Woodruff (2007) document that a one-standard deviation improvement in the quality of the legal system is associated with a $0.15 - 0.30$ standard deviation increase in firm size.

Parts (ii) shows that when socially optimal contracts are interior (i.e., $\widehat{\pi}^S < \pi^S_{\max}$), the financing friction by itself does not create any inefficiencies. This holds when the supplier's rents turn out to be large enough for the financing constraint to be slack. It is worth noting, however, that the result does not say that investment is at its first-best level. This is in sharp contrast to standard results in the literature—for instance, Proposition 1 by Thomas and Worrall (1994)—in which interior socially optimal contracts only occur when investment is efficient. This result is true in my model thanks to the combination of two assumptions: the buyer's ability to renege on ex post payments and the fact that he is financially unconstrained.

Finally, part (iii) shows that if there is a unique corner socially optimal contract, investment distortions arise both due to both the financing constraint and the supplier's non-reneging constraints. This will happen when even under the supplier's profit maximizing equilibrium, rents to the latter are relatively small. The financing constraint thus remains a concern and the initial payment is as large as allowed by the supplier's non-reneging constraints.

The previous result shows that the financing friction affect the long-run level of social welfare only when there is a unique corner socially optimal contract. A natural question is then under what conditions this is true. The next proposition offers an answer.

Proposition 8.4 *There exists a* $\underline{\phi} \in (0,1)$ *such that* $\widehat{\pi}^S < \pi^S_{\max}$ *for any* $\phi < \underline{\phi}$. *Moreover, for any* $\phi \geq \underline{\phi}$, *there is a* $\delta_\phi \in (0,1)$ *such that* $\widehat{\pi}^S = \pi^S_{\max}$ *for* $\delta \leq \delta_\phi$ *and* $\widehat{\pi}^S < \pi^S_{\max}$ *for* $\delta > \delta_\phi$.

When the financing friction is small, the supplier can by himself cover most of the investment cost. The initial transfer needed by the supplier for the financing constraint not be a concern is small and it would eventually be reached. Thus socially optimal contracts are interior. When the financing friction is large, on the other hand, socially optimal contracts can be either in a corner (i.e., $\widehat{\pi}^S = \pi^S_{\max}$) or interior (i.e., $\widehat{\pi}^S < \pi^S_{\max}$), depending on the discount factor. Intuitively, when ϕ is high, the supplier must be getting large rents for the financing constraint not to be a concern and thus for socially optimal contracts to be interior. This, in turn, would be true when the value of the relationship is also large, which occurs when δ is high and the inefficiencies caused by the enforcement friction are not too severe.

Finally, I conclude this section by discussing one simple way of implementing any socially optimal contract.

Proposition 8.5 (Implementation of Optimal Contracts)

(i) The buyer's most preferred socially optimal contract, i.e., $\left(\widehat{\pi}^S, Q\left(\widehat{\pi}^S\right)\right)$, *can be implemented by a sequence of stationary contracts in which*

$$l\left(\widehat{\pi}^S\right) = \phi c k^o, \textit{and } T\left(\theta; \widehat{\pi}^S\right) = \max\left\{\beta^s R\left[\left(\widehat{\pi}^S\right), \theta\right] + A, 0\right\}, \forall \theta \in \Theta,$$

for some $A \in \left(-R\left(k^o, n\right), \pi^J\left(\widehat{\pi}^S\right) - \widehat{\pi}^S - \underline{\pi}^B\right)$.

(ii) Any agreement achieving $\pi^S \in \left(\widehat{\pi}^S, \pi^S_{\max}\right]$ *can be implemented by a sequence of stationary contracts in which:*

$$l\left(\pi^S\right) = \phi c k^o + \frac{\Delta \pi^S}{1-\delta}, \textit{ and } T\left(\theta; \pi^S\right) = T\left(\theta; \widehat{\pi}^S\right) - \frac{\delta}{1-\delta}\Delta\pi^S, \forall \theta \in \Theta,$$

where $\Delta\pi^S := \pi^S - \widehat{\pi}^S$.

The buyer makes two fixed payments: one up front, before investment decisions are made, and one ex post, when the components are delivered (which can be either positive or negative). The supplier also receives a share β^s of the total revenue when sufficient resources are available. Three features of this implementation are worth noting. First, when the supplier's profits are not too large, there is co-investment, i.e., $l\left(\pi^S\right) < ck^o$. In this case and provided that $A < 0$, the contract has an interesting interpretation: it resembles a *joint venture agreement* in which the buyer covers part of the investment cost in exchange of debt with face value A and a fraction $1 - \beta^s$ of the project's shares. Second, when the supplier's equilibrium profits rise, the initial transfer increases and ex post payments decrease. As a consequence, the supplier's profits become less dependent on the profitability of the project. He is thus treated less as a partner and more like just an external supplier, who charges the final price of the components in advance. And third, when $\pi^S > \widehat{\pi}^S$, it is possible for ex post transfers to be negative, i.e., $T(\theta)\left(\pi^S\right) < 0$. This can be thought of as a payback agreement. The buyer contributes a larger share of the investment cost than is required by the supplier's financing constraint. In exchange, he demands compensation when market conditions turn out to be extremely bad.[14]

8.3 Welfare and the Financing Friction

In this section, I study in more detail how the level of social welfare (i.e., joint profits) achieved by socially optimal contracts is affected by variations in the magnitude of the financing friction. To develop some intuition, consider two scenarios with $\phi_1 = 0$ and $\phi_2 \in (0,1)$, respectively. Because a higher ϕ forces the buyer to make a larger initial payment, the supplier's short-term payoffs from reneging on the agreement increase. One could then expect it to be harder for firms to sustain a high investment level when $\phi = \phi_2 \in (0,1)$. The previous argument makes, however, the implicit assumption that the long-term gains from reneging are the same, which turns out to be false. As shown by Proposition 5.3, the financing friction also affects the firms' payoffs under spot transaction. This generates a second effect. If the firms' payoffs under spot transaction decrease when the supplier's financing condition worsen, the incentives to respect the agreement are strengthened. Depending on which of these two effects dominates, social welfare can increase or decrease with ϕ. The next result is proved in the Appendix.

Proposition 8.6 (Social Welfare and the Financing Friction) *Let $\pi^J(\phi)$ denote the firms' expected joint profits under a socially optimal contract when the size of the financing friction is ϕ, i.e., $\pi^J(k^o(\phi)) := -ck^o(\phi) + \mathbb{E}\left[R(k^o(\phi), \theta)\right]$.*

[14] As in Proposition 7.1, this implementation is not unique, not even within the class of stationary contracts. For example, when the investment level is efficient and the buyer's non-reneging constraints are all slack, the buyer can also increase the supplier's share of the total surplus by raising the initial payment and leaving ex post payments unchanged.

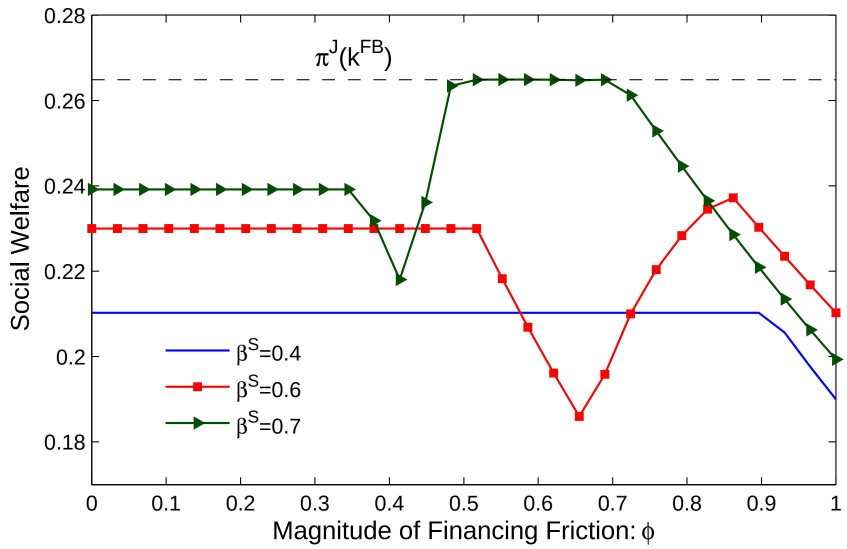


Figure 5: Investment under Socially Optimal Contracts as function of β^S

(i) If socially optimal contracts are interior (i.e., $\pi^S_{\max} < \widehat{\pi}^S$), then $\pi^J(\phi)$ is independent of ϕ.

(ii) If there is a unique corner socially optimal contract (i.e., $\pi^S_{\max} = \widehat{\pi}^S$), then there exist values ϕ_1, ϕ_2, and ϕ_3, with $0 < \phi_1 \leq \phi_2 \leq \phi_3$, such that $\partial\pi^J/\partial\phi < 0$ for $\phi < \phi_1$ and $\partial\pi^J/\partial\phi > 0$ for $\phi \in (\phi_2, \phi_3)$.

The intuition for the first part is simple. When $\pi^S_{\max} < \widehat{\pi}^S$, the discussion of Proposition 8.3 suggests that the financing friction is not a concern. An increase in ϕ, therefore, has no impact on the investment level and social welfare is unchanged. As ϕ keeps increasing, however, Proposition 8.4 implies that eventually there will be a unique corner socially optimal contract ($\pi^S_{\max} = \widehat{\pi}^S$) and changes in ϕ will have an effect on welfare. The key insight of the second part of the proposition is that the negative effect of an increase in ϕ dominates when ϕ is relatively small and the positive effect dominates for intermediate values of ϕ. The intuition behind this result is as follows. When ϕ is initially small, increasing ϕ raises the supplier's long-term profits in case of a deviation (see Proposition 5.3). Although the buyer's long-term deviation profits decrease, the sum of the two negative effects, higher profits for the supplier in case of a deviation and the need of a larger initial payment to sustain the original investment level, dominates the marginal benefit derived from the decrease in the buyer's deviation profits. When ϕ is relatively large, on the other hand, increasing ϕ reduces both firms' deviation profits. If, in addition, ϕ is not too large, the original initial payment is relatively small and can be increased at low cost (in terms of incentives). In this case, the marginal benefit of an increase in ϕ dominates the marginal costs.

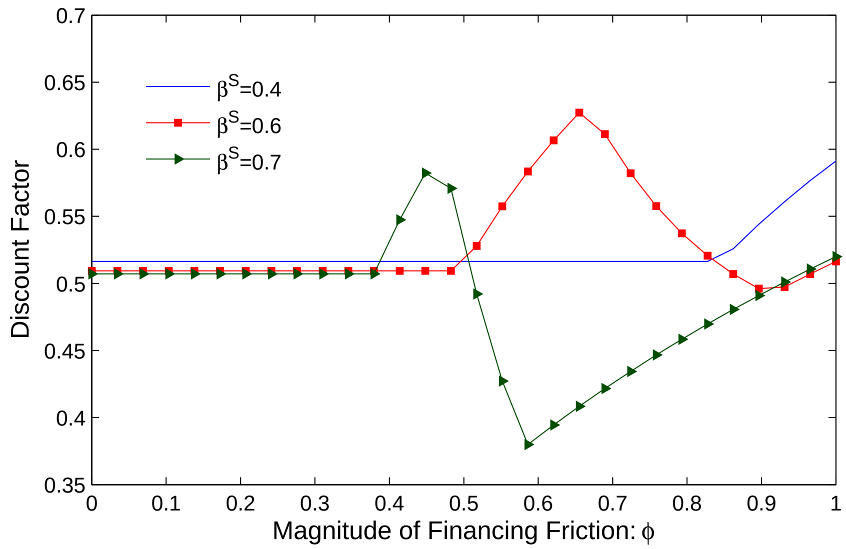


Figure 6: Minimum Discount Factor that Implements k^{FB} as function of ϕ

An interesting question concerns the "optimal" level of the financing friction. That is, the value of ϕ under which social welfare is maximized. Proposition 8.6 implies that depending on the parameter of the model, social welfare can decrease, increase, or not change when the financing friction rises. It is not, therefore, entirely clear whether a given positive degree of financial frictions is detrimental or beneficial to social welfare relative to the case in which credit market are perfect ($\phi = 0$). The next proposition shows that the answer turns out to depend critically on the allocation of bargaining power.

Proposition 8.7 *If the supplier's bargaining power is relatively large (i.e., $\beta^s > 1/2$), a certain degree of imperfections in financial markets (i.e., $\phi > 0$) can increase social welfare relative to the case of perfect credit markets. If the supplier's bargaining power is relatively small (i.e., $\beta^s \leq 1/2$), on the other hand, any positive degree of financing frictions is (weakly) detrimental to social welfare.*

To illustrates the results of Propositions 8.6 and 8.7, figure 5 depicts the level of social welfare achieved by socially optimal contracts as a function of the magnitude of the financing friction. For all numerical computations, I assume that the revenue function is given by $R(k, \theta) = \theta k^{.65}$, where $\theta \in \{.8, 1, 1.2\}$ and $f_\theta = 1/3$ for all θ, that the per unit cost of capacity c equals 1, and that the discount factor δ equals 0.6. The dashed line corresponds to first-best level of social welfare. The figure shows that when the supplier's bargaining power is relatively small, ϕ_2 may not exist. Welfare is then independent of ϕ when the latter is small and strictly decreasing when ϕ is close to one. If β^s is relatively large, on the other hand, both ϕ_2 and ϕ_3 exists and welfare is U-shaped for intermediate values of

ϕ: it first decreases but then increases. As suggested by Proposition 8.7, social welfare is not maximized when credit markets are perfect but instead for intermediate values of ϕ. Moreover, Figure 5 shows that when the supplier's bargaining power is relatively large, the first-best level of social welfare can actually be reached for intermediate values of ϕ but not when $\phi = 0$. The figure also suggest that the higher the supplier's bargaining power, the bigger the welfare gain from having a certain degree of imperfections in credit markets.

Figure 6 offers an alternative way to interpret Propositions 8.6 and 8.7. Define δ^* as the minimum value of the discount factor needed for socially optimal contracts to implement the first-best level of social welfare. Remember from Proposition 4.1 that absent any enforcement problem, investment (and thus social welfare) is at its first-best level regardless of the value of ϕ. One can therefore think of the value of δ^* as a measure of the inefficiencies caused by contracting frictions. The figure plots the value of δ^* as a function of the magnitude of the financing friction. The figure shows that an increase in ϕ can decrease δ^*. Thus, as suggested by the previous discussion, the welfare loss introduced by weak enforcement can decrease with the magnitude of the financing frictions.

8.4 Other Efficient Contracts

As mentioned in section 8.2, when the supplier's continuation profits are small (i.e., $\pi^S < \widehat{\pi}^S$), efficient self-enforcing contracts are not socially optimal. Here I analyze such contracts. The first Lemma will be useful to understanding the intuition behind the rest of the section.

Lemma 8.8 *The following is true on the interval* $\left[\pi^S_{\min}, \widehat{\pi}^S\right]$*:*

(i) The supplier's financing constraint (F-C) is binding, i.e., $l = \phi c k$*.*

(ii) The function Q *is strictly concave.*

Part (i) establishes the key role played by the supplier's financing constraint and confirms Proposition 7.1: when financing constraints are not a concern, all efficient contracts achieve the same level of social surplus. The second part strengthens Lemma 8.1 above by showing that when the supplier's expected discounted profits are not large enough to sustain a socially optimal contract, the concavity of the value function is strict. The strict concavity of Q will be useful to understanding the efficient choice of transfers and continuation profits.

The following result shows that as long as efficient contracts are not optimal, they must have a *bang-bang* structure.

Proposition 8.9 (Current Transfers) *For all* $\theta \in \Theta$*, if* $T(\theta) > 0$ *then* $\pi^S(\theta) \geq \widehat{\pi}^S$*.*

In other words, when $\pi^S < \widehat{\pi}^S$ and as long as $\pi^S(\theta) < \widehat{\pi}^S$ (i.e., the next period continuation contract remains non-optimal), the supplier earns no current rents. Instead, he is incentivised solely through the promise of higher profits in the future—adjustments in future continuation profits. Intuitively, since both firms are risk neutral and share the same discount factor, the specific timing of payments is irrelevant provided that its expected discounted value remains unchanged. But delaying compensation has an advantage. It allows the buyer to offer a "carrot" to the supplier which can be reused for many periods to prevent the latter from reneging on the agreement, thus constituting the most efficient mean of incentive provisions. Once $\pi^S(\theta)$ reaches a value larger than $\widehat{\pi}^S$, the cost of keep delaying payments, which is given by the slope of Q, turns out to be the same as the cost of current compensation (i.e., -1) and positive transfers may occur.

Next I turn to the optimal choice of continuation profits.

Proposition 8.10 (Continuation Profits) *If $\pi^S < \widehat{\pi}^S$, then $\pi^S(\theta) \geq \pi^S$ for all $\theta \in \Theta$, with at least one strict inequality.*

Put another way, future continuation profits never decrease and, with probability one, increase for at least one realization of market conditions. To further understand Proposition 8.10, note that since Q is strictly concave by Lemma 8.8, setting $\pi^S(\theta)$ different from π^S is costly to the buyer. The reason why continuation profits never decrease is then closely connected to the intuition of part (i) in Proposition 8.2: it is costly due to the concavity of Q but does not offers extra benefits in terms of relaxing any of the constraints. When continuation profits drop below its current level (i.e., $\pi_2^S(\theta) < \widehat{\pi}^S$), it must be because otherwise the buyer would have incentives to breach the contract and offer a lower payment for the components (constraint NR-B$_\theta$ must be binding). But the only reason why the buyer may want to renege is because the original payment is too "large". This in turn implies that the same incentives can be provided in a more efficient manner by decreasing $T(\theta)$ instead. In contrast, as mentioned in the discussion of the previous proposition, delaying compensation has the benefit of decreasing the supplier's temptation to renege on the contract for more than one period. The result then states that sometimes such benefit turns out to be large enough to more than compensate the cost associated with the strict concavity of Q.

I now present the two main results of this section. Theorem 8.11 shows that any efficient self-enforcing contract must exhibit a continuation that maximizes social welfare (joint profits) among all such self-enforcing contracts. Theorem 8.12, on the other hand, offers some insights into the behavior of the optimal investment policy

Theorem 8.11 *Let π_t^S stands for the supplier's expected continuation profits at period t. For any efficient relational contract $\psi \in \Psi^*$, with probability one, there exists a random time*

$T(\psi) < \infty$ such that $\pi^S_{T(\psi)} \geq \widehat{\pi}^S$.

Proposition 8.10 suggests that starting from any point on the Pareto frontier, equilibrium payoffs will increase and eventually reach the socially optimal region. Theorem 8.11 shows that this is true and that convergence must occur within a *finite number of periods*. The basic idea behind this result—that when agents have the opportunity to renege on their promises, optimal incentives provision involves backloading compensation—goes back to Becker and Stigler (1974) and Harris and Holstrom (1982). More recently, Ray (2002) derives a similar conclusion under very general conditions. My model, however, differs from Ray's in two important respects. First, either firm can repudiate his obligations by exploiting his bargaining power when the components are delivered. This situation is ruled out by Ray who assumes that one of the parties can commit to the current agreement (provided that an interim participation constraint is satisfied). In addition, he also assumes that when the terms of the agreement change, agents cannot do better by deviating than they can do by adhering to the agreement. This does not hold in the current set-up. If the initial payment is "low enough", a higher l may increase the supplier's deviation payoffs more than it increases his payoffs if he stays in the relationship.

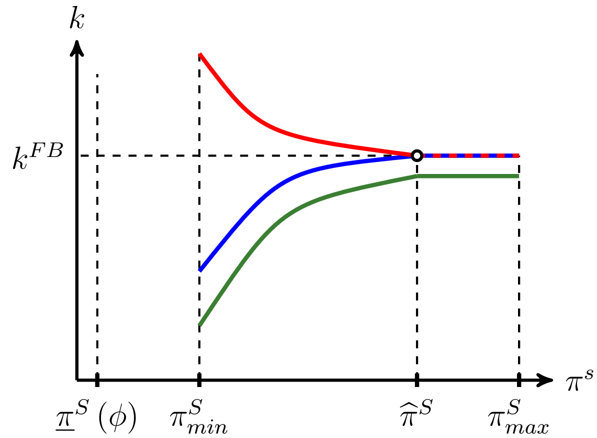
Theorem 8.12 (Investment) *Let $\pi^S < \widehat{\pi}^S$. The following is true in any efficient relational contract:*

(i) The investment level $k\left(\pi^S\right)$ is singled-valued and continuous in π^S.

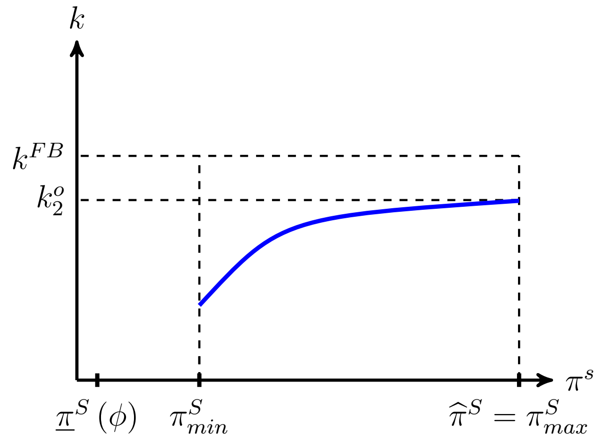
(ii) Investment is inefficient, i.e., $k\left(\pi^S\right) \neq k^{FB}$, and can be either above or below its first-best level.

(iii) Investment distortions, i.e., $|k\left(\pi^S\right) - k^{FB}|$, are strictly decreasing in π^S.

The first part of the proposition states that the efficient investment policy is unique and continuous. This follows directly from the strict concavity of the objective function (Lemma 8.8). Part (ii) shows that if the supplier's profits are sufficiently low, investment decisions are distorted. Economists have long been concerned with problems of under-investment and hold-up. Interestingly, Theorem 8.12 shows that weak contract enforcement, coupled with financing constraints, can lead to the opposite outcome: over-investment. The intuition for this result is as follows. As mentioned earlier, by changing the initial payment, the parties can redistribute surplus without affecting incentives to honor the agreement. In the current set up, however, the liquidity and financing constraints impose a lower bound on the initial transfers. The previous method may then fail when trying to increasing the buyer's profits. In this context and as long as the fraction of the cost covered by the buyer is small, more investment allows the latter to extract rents from the supplier without affecting the ex post



(A) Interior Optimal Contracts



(B) Corner Optimal Contract

Figure 7: Investment Policy as a function of π^S

non-reneging constraints. In other words, investment has an additional value as it may also be used to simulate an up front payment from the supplier when transfers to the latter are already minimal. There is of course an efficiency loss in over-investing. Yet, it may be the only way to redistribute surplus from the supplier to the buyer. Note that over-investment is not the result of agency problems between the manager of a firm and its shareholders. In fact, the preferences between the two are perfectly aligned. The model therefore provides a complementary explanation to one of the most well established stylized facts in the corporate finance literature: the tendency of managers to over-invest; see, e.g., Stein (2003).

Finally, the last part of the proposition shows that as the supplier's continuation profits increase, inefficiencies decrease and investment monotonically approaches its first-best level. An intuitive argument is as follows. Investment distortions are caused by the firms' inability to commit and the supplier's weak financial conditions. But as π^S increases, the supplier's temptation to renege on the contract decreases. In addition, because he must be receiving larger transfers (in the form of both a larger initial payment and larger ex post compensation), his liquidity problems becomes less severe (i.e., the financing and liquidity constraints become less binding). These two effects, in turn, ensure that investment decisions becomes less distorted. Note that because the function Q is strictly decreasing, a higher π^S also implies lower profits to the buyer. One could then wonder whether his incentives to renege on the agreement increase. And if they do, why this does not negatively affect investment decisions. Interestingly, because the buyer is not liquidity constrained, it is always possible to use transfers to increase the supplier's profits without affecting the buyer's incentives to breach the contract. The different paths of investment are illustrated in 7.

To complete this section, it is worth noting that the facts that the supplier's future continuation profits never drop below its current level and that investment distortions strictly decrease with π^S imply the following result regarding the evolution of investment over time.

Corollary 8.13 (Investment Dynamics) *In any efficient relational contract in which the supplier's profits are relatively small (i.e., $\pi^S < \widehat{\pi}^S$), investment distortions (i.e., $|k\left(\pi^S\right) - k^{FB}|$) are decreasing over time. After a finite number of periods, investment converges to a constant level.*

9 Conclusions

This paper explores the interaction between contractual frictions and imperfections in financial markets. In particular, I study how the efficiency of informal agreements is affected when one of the parties has limited access to credit markets. I develop a theoretical framework to address this issue and use recursive methods to provide a full characterization of the properties of Pareto efficient self-enforcing agreements. The analysis suggests that a certain level of frictions in financial markets can improve the degree of relational contracting. The reason is that when two firms with different financial conditions engage in an informal agreement, inefficient financial markets can increases the "relationship collateral". Intuitively, the firm with better access to credit acts as a bank for the other firm, which would otherwise have limited access to financing. This increases both firms' incentives to honor the agreement by raising the value of the relationship relative to the case of spot transactions.

The model also predicts that, under some conditions, young partnerships may feature inefficiently high investment levels. Interestingly, over-investment is an efficient response to an institutional environment characterized by the two previously mentioned frictions, not the result of agency problems between the manager of a firm and its shareholders. In fact, in the current set up, the preferences between the two are perfectly aligned. The model therefore provides a complementary explanation to one of the most well established stylized facts in the corporate finance literature: the tendency of managers to over-invest (see, e.g., Stein (2003)).

Although the model does not explicitly consider the possibility of in-house production, the mechanism linking financing frictions and the value of relational contracting would remain valid if the buyer had the option of producing the component himself. This suggests that extending the model along these lines can help to understand the relationship between financial development, contractual imperfections, and the internal organization of firms (i.e., vertical integration). In such an extension, a firm may decide to outsource in the presence of financial frictions, but would choose in-house production when financial markets are efficient. Thus, better financial markets may lead to more vertical integration. Interestingly, this accords well with recent empirical evidence by Acemoglu, Johnson, and Mitton (2009), who find that the correlation between contracting costs and vertical integration is not very robust. Instead, the authors argue that vertical integration is more likely in countries that have *both* greater contractual costs and better financial markets.

10 Appendix

11 Proofs of Section 4

Proof of Proposition 4.1. I first show that there is no under-investment, i.e., $k_t \geq k^{FB}$ for all $t \geq 1$. Suppose to the contrary that there exists a Pareto efficient contract and a period t such that $k_t < k^{FB}$. Consider a small increase in k_t of $\Delta k > 0$, an increase in l_t of $\Delta l = \phi c \Delta k$, and an increase in $T_t(\theta)$ of $\Delta T_t(\theta) = -\phi c + \mathbb{E}\left[R_k(k_t, \theta)\right]$ for all $\theta \in \Theta$. Note that $\Delta T(\theta) > -c + \mathbb{E}\left[R_k(k_t, \theta)\right] > 0$, where the first inequality follows from the definition of $\Delta T(\theta)$ and the second inequality from the fact that, by assumption, investment is inefficient. Thus $\Delta T(\theta)$ is indeed positive. It is immediate to verify that the new contract is feasible, that the buyer's profits remain unchanged, and that the supplier's expected profits at time t increase by $\Delta\pi_t^S \approx \Delta k(1-\delta)\{-c + \mathbb{E}\left[R_k(k_t, \theta)\right]\} > 0$, where the inequality holds since $k_t < k^{FB}$. But this implies that the original contract is not efficient, a contradiction.

Next I show that there is no over-investment, i.e., $k_t \leq k^{FB}$ for all $t \geq 1$. Suppose by contradiction that there is a period t such that $k_t > k^{FB}$ and consider the following perturbation to the original contract: a small decrease in k_t of $\Delta k > 0$ and a decrease in l_t of $\Delta l = \phi c \Delta k$. It is straightforward to verify that the financing and the liquidity constraints are both satisfied and that the buyer's and the supplier's expected profits at time t change by $\Delta\pi_t^B \approx \Delta k\{\phi c - \mathbb{E}\left[R_k(k_t, \theta)\right]\}$ and $\Delta\pi_t^S = \Delta k(1-\phi)c$, respectively. Since, by assumption, the original contract features positive investment, there must be a period and a demand realization in which the supplier receives strictly positive rents (i.e., constraint (L-C$_\theta$) is slack), otherwise his discounted expected profits would be negative and the promise-keeping condition would be violated. Let t' and θ' stand for such period and demand realization, respectively, and consider decreasing $T_{t'}(\theta')$ by $\Delta T_t(\theta) = \delta^{t-t'}\Delta k\{\phi c - \mathbb{E}[R_k(k_t,\theta)]\}$. Observe that the change is valid for sufficiently small Δk. Moreover, after some calculations, one can show that the buyer's profits remain the same as under the original contract and that the supplier's profits at time 1 change by $\Delta\pi_1^S = \delta^{t-1}\Delta k\{c - \mathbb{E}[R_k(k_t,\theta)]\} > 0$, where the inequality follows since $k_t > k^{FB}$. But this contradicts the optimality of the original contract. ■

12 Proofs of Section 5

Proof of Proposition 5.1. *Part (i).* After substituting (5.2) into (5.3), the buyer's problem becomes

$$\max_{l\geq 0}\begin{cases} -l + (1-\beta^s)\,\mathbb{E}[R(l/(\phi c),\theta)] & \text{if } l \leq \phi c\underline{k}_u^* \\ -l + (1-\beta^s)\,\mathbb{E}[R(\underline{k}_u^*,\theta)] & \text{if } l > \phi c\underline{k}_u^*. \end{cases}$$

It is immediate from the previous expression that $l \leq \phi c\underline{k}_u^*$, otherwise the buyer could reduce l without affecting the level of capacity, strictly increasing his profits. The problem can then be written as

$$\max_{l\geq 0} \; -l + (1-\beta^s)\,\mathbb{E}[R(l/(\phi c),\theta)]$$

subject to $l \leq \phi c\underline{k}_u^*$. Setting the first-order condition equal to zero yields

$$-1 + \frac{1-\beta^s}{\phi c}\mathbb{E}[R_k(l/(\phi c),\theta)] + \mu^+ - \lambda = 0, \tag{12.1}$$

where μ^+ and λ are the Khun-Tucker multipliers associated with $l \geq 0$ and $l \leq \phi c\underline{k}_u^*$, respectively. Now consider the following cases depending on the value of λ.

Case 1: Constraint $l \leq \phi c\underline{k}_u^*$ is slack, i.e., $\lambda = 0$. It is straightforward to check when $\lambda = 0$, (12.1) coincides with the first-order condition associated with (5.4). Thus $k = \underline{k}_B^*$ and $l = \phi c\underline{k}_B^*$. Furthermore, the assumption that the constraint is slack will be true if and only if $\underline{k}_B^* \leq \underline{k}_u^*$.

Case 2: Constraint $l \leq \phi c\underline{k}_u^*$ is binding so that $l = \phi c\underline{k}_u^*$ and $k = \underline{k}_u^*$. It is only left to show that the constraint will be binding if and only if $\underline{k}_B^* > \underline{k}_u^*$. Using (12.1), $\lambda > 0$ if an only if $-1 + (1-\beta^s)(\phi c)^{-1} E[R_k(\underline{k}_u^*,\theta)] + \mu^+ > 0$ or equivalently, if and only if $\underline{k}_B^* > \underline{k}_u^*$.

Part (ii). To prove that investment is strictly inefficient, consider first the case in which $\beta^s < 1$. The result then holds since $\underline{k}^* = \min(\underline{k}_B^*, \underline{k}_u^*) \leq \underline{k}_u^* < k^{FB}$, where the last inequality uses the definition of $\underline{k}_u^*$. If $\beta^s = 1$, on the other hand, it is easy to verify that $\underline{k}_B^* = 0$ so that $\underline{k}^* = \underline{k}_B^* = 0$ by part (i). ■

Proof of Proposition 5.2. First note that the assumptions on the revenue function R rule out corner solutions. Using (5.4) and (5.1), $\underline{k}_B^*$ and $\underline{k}_u^*$ are then implicitly defined by the following first-order conditions:

$$\mathbb{E}\left[R_k\left(\underline{k}_B^*, \theta\right)\right] = \frac{\phi c}{1-\beta^s}, \tag{12.2}$$

$$\mathbb{E}\left[R_k\left(\underline{k}_u^*, \theta\right)\right] = \frac{c}{\beta^s}. \tag{12.3}$$

Using (12.2), (12.3), together with the strict concavity of R with respect to k, it is immediate to verify that

$$\frac{\partial \underline{k}_B^*}{\partial \beta^s} < 0,\ \frac{\partial \underline{k}_B^*}{\partial \phi} < 0,\ \frac{\partial \underline{k}_u^*}{\partial \beta^s} > 0,\ \text{and}\ \frac{\partial \underline{k}_u^*}{\partial \phi} = 0. \tag{12.4}$$

Part (i). Since $R_{kk} < 0$, $\underline{k}_B^* \leq \underline{k}_u^*$ if and only if $\phi c \left(1-\beta^s\right)^{-1} \geq c\left(\beta^s\right)^{-1}$ or equivalently, $\beta^s \geq 1/\left(1+\phi\right)$. Thus $\underline{k}_B^* \leq \underline{k}_u^*$ for $\beta^s \in \left[1/\left(1+\phi\right), 1\right)$ and $\underline{k}_B^* \geq \underline{k}_u^*$ for $\beta^s \in \left[0, 1/\left(1+\phi\right)\right)$. The result then follows from (12.4) and the definition of $\underline{k}^*$, i.e., $\underline{k}^* = \min\left(\underline{k}_B^*, \underline{k}_u^*\right)$.

Part (ii). That $\underline{k}^*$ is weakly decreasing in ϕ holds since $\underline{k}_u^*$ does not depend on ϕ and $\underline{k}_B^*$ is decreasing in ϕ. For the second claim, I will show that $\underline{k}^* = \underline{k}_u^*$ for all $\phi \in [0, \widehat{\phi}\,]$ and that $\underline{k}^* = \underline{k}_B^*$ for all $\phi \in [\,\widehat{\phi}, 1]$. The result then follows from (12.4). Given the definition of $\underline{k}^*$, to establish that $\underline{k}^* = \underline{k}_u^*$ it suffices to show that $\underline{k}_B^* \geq \underline{k}_u^*$. From the proof of part (i), $\underline{k}_B^* \geq \underline{k}_u^*$ if and only if $\left(1-\beta^s\right)^{-1} \leq c\left(\beta^s\right)^{-1}$ or equivalently, $\phi \leq \left(1-\beta^s\right)/\beta^s$. Thus, if $\left(1-\beta^s\right)/\beta^s \geq 1$, then $\underline{k}^* = \underline{k}_u^*$ for all ϕ. Alternatively, if $\left(1-\beta^s\right)/\beta^s < 1$, then $\underline{k}^* = \underline{k}_u^*$ for all $\phi \leq \left(1-\beta^s\right)/\beta^s$ and $\underline{k}^* = \underline{k}_B^*$ for all $\phi > \left(1-\beta^s\right)/\beta^s$. ■

Proof of Proposition 5.3. *Part (i).* Note that $\underline{\pi}^S + \underline{\pi}^B = -ck + E\left[R\left(k, \theta\right)\right]$. Thus joint profits only depend on k and the result then follows directly from part (i) of Proposition 5.2.

Part (ii). Recall that $\underline{\pi}^B = -\phi c \underline{k}^*\left(\phi\right) + \left(1-\beta^s\right) E\left[R\left(k, \theta\right)\right]$. Now consider increasing ϕ by a small $\Delta\phi > 0$. By Proposition 5.2, $\underline{k}^*$ is decreasing in ϕ and there are two cases to consider. First, it should be clear that if capacity remains the same, i.e., $\underline{k}^*\left(\phi + \Delta\phi\right) = \underline{k}^*\left(\phi\right)$, then $\Delta\underline{\pi}^B \approx -\Delta\phi c \underline{k}^*\left(\phi\right) < 0$. Alternatively, if $\underline{k}^*\left(\phi + \Delta\phi\right) < \underline{k}^*\left(\phi\right)$, then the fact that $\underline{k}_u^*$ is independent of ϕ implies $\underline{k}^*\left(\phi + \Delta\phi\right) = \underline{k}_B^*\left(\phi + \Delta\phi\right)$. Moreover, since $\underline{k}^*$ is continuous in ϕ, $\underline{k}^*\left(\phi\right) = \underline{k}_B^*\left(\phi\right)$ for small enough $\Delta\phi$. That $\Delta\underline{\pi}^B < 0$ then follows since $\max_{k\geq 0}\left\{-\phi ck + \left(1-\beta^s\right)\mathbb{E}\left[R\left(k, \theta\right)\right]\right\}$ is strictly decreasing in ϕ.

Part (iii). By Proposition 5.2, k is constant for all $\phi \in [0, \widehat{\phi}\,]$. Thus, using Proposition 5.1, increasing ϕ by a small $\Delta\phi > 0$ changes the supplier's profits by $\Delta\underline{\pi}^S \approx \Delta\phi c \underline{k}^*\left(\phi\right) > 0$. For $\phi \in [\,\widehat{\phi}, 1]$, the proof of part (iii) of Proposition 5.2 implies that $\underline{k}^* = \underline{k}_B^*$. Differentiating $\underline{\pi}^S$ with respect to ϕ then yields

$$\frac{\partial \underline{\pi}^S}{\partial \phi} = c\underline{k}_B^* + \frac{\partial \underline{k}_B^*}{\partial \phi}\left\{-\left(1-\phi\right)c + \beta^s \mathbb{E}\left[R_k\left(\underline{k}_B^*, \theta\right)\right]\right\}$$

or, using (12.2) and by some algebra,

$$\frac{\partial \underline{\pi}^S}{\partial \phi} = c\underline{k}_B^* \left[1 + \epsilon_k(\phi) \left(-\frac{1}{\phi} + \frac{1}{1-\beta^s}\right)\right],$$

where $\epsilon_k(\phi) := \phi \frac{\partial \log \underline{k}_B^*(\phi)}{\partial \phi}$. Since $c\underline{k}_B^* > 0$ (remember from the proof of Proposition 5.2 that $\underline{k}_B^* > 0$), it suffices to prove that

$$1 + \epsilon_k(\phi) \left(-\frac{1}{\phi} + \frac{1}{1-\beta^s}\right) < 0.$$

To this end note that differentiating (12.2) with respect to ϕ and using the definition of $\epsilon_k(\phi)$ yields

$$\epsilon_k(\phi) = \mathbb{E}\left[\frac{1}{\underline{k}_B^*} \frac{R_k(\underline{k}_B^*, \theta)}{R_{kk}(\underline{k}_B^*, \theta)}\right],$$

which, by Assumption 1, ensures that $\epsilon_k(\phi) < -1$ for all θ.

In addition, let $g(\phi, \beta^s) := \left(-\frac{1}{\phi} + \frac{1}{1-\beta^s}\right)$ and observe that

$$\begin{aligned} g\left(\widehat{\phi}, \beta^s\right) &= \left[-\frac{1}{\min\left(1, \frac{1-\beta^s}{\beta^s}\right)} + \frac{1}{1-\beta^s}\right] \\ &= \max\left[\frac{\beta^s}{1-\beta^s}, 1\right] \\ &\geq 1. \end{aligned}$$

The result then follows since $\partial g / \partial \phi > 0$ and , as showed before, $\epsilon_k(\phi) < -1$ for all θ. ■

13 Proofs of Section 6

Proof of Proposition 6.1. The necessity of (6.1) and (6.2) follows from the ability of the buyer to refuse to make the initial payment and the ability of the supplier to choose any level of investment, respectively, and then permanently revert to spot transaction. The necessity of (6.3) and (6.4) follows since, at any point in time, either party can choose to use their ex post bargaining power and then permanently terminate the relationship. Finally, (6.5) and (6.6) are simply the financing and liquidity constraints, respectively

To prove sufficiency, consider an allocation satisfying (6.1)-(6.6) and construct the following off-equilibrium strategies: any deviation results in permanent reversion to the static Nash equilibrium (spot transactions). Since payoffs are continuous at infinity (a condition that essentially requires actions in the far future to have a negligible impact on current payoffs), it is possible to restrict attention to one-shot deviations; see Mailath and Samuelson (2006). For deviations in which the buyer offers an initial payment that is different from the one specified by the equilibrium, it is easy to verify that—since after observing the deviation the supplier will respond by permanently reverting to spot transactions—the best deviation for the buyer is precisely the initial transfer under spot transactions. But (6.1) implies that such deviation is not profitable. Similarly, if the supplier decides to invest an amount that is not the one specified by the equilibrium, his most profitable deviation generates current payoffs of $\pi^S_{spot}\left[l_t\left(h^t_e\right)\right]$ followed by perpetual spot transactions. But again, (6.2) ensures that this would not increase the supplier profits. Finally, if either firm deviates from the equilibrium voluntary ex post transfers, then the other party will respond by using his bargaining power in the current period, delivering a payment of $\beta^s R(k,\theta)$ to the supplier and $(1-\beta^s)R(k,\theta)$ to the buyer, followed by spot transactions forever after. Equations (6.3) and (6.4) guarantee that no firm benefits from such deviation. Lastly, (6.5) and (6.6) make sure that the equilibrium is feasible for all periods and potential histories of demand realizations. ■

Proof of Lemma 6.3. Let $\pi^S_{\min}$ be the infimum over $\left[\underline{\pi}^S, \pi^S_{\max}\right]$ for which constraint (P-K) is binding. I first show that (P-K) must be binding for any $\pi^S > \pi^S_{\min}$. The result is trivial when $\pi^S_{\min} = \pi^S_{\max}$ so let $\pi^S_{\min} < \pi^S_{\max}$. Suppose by contradiction that there is a $\pi^S > \pi^S_{\min}$ for which (P-K) is slack. Since the problems for $\pi^S_{\min}$ and π^S are identical except for the fact that the former has an additional binding constraint, it must be that $Q\left(\pi^S_{\min}\right) < Q\left(\pi^S\right)$. But also note that the solution for π^S satisfies all the constraints for $\pi^S_{\min}$, implying that $Q\left(\pi^S_{\min}\right) \geq Q\left(\pi^S\right)$. Hence a contradiction. To prove that $Q'\left(\pi^S\right) = 0$ for any $\pi^S \in \left(\underline{\pi}^S, \pi^S_{\min}\right)$, first observe that ignoring the promise-keeping condition, the solution to Problem (P1) is independent of the value of π^S. Furthermore, by the definition of $\pi^S_{\min}$, (P-K) is slack for all $\pi^S < \pi^S_{\min} = \pi^S_{\max}$. By the continuity of Q, it then follows that $Q\left(\pi^{S'}\right) = Q\left(\pi^{S''}\right)$ for any $\pi^{S'}, \pi^{S''} \in \left[\underline{\pi}^S, \pi^S_{\min}\right]$. Finally, I argue that $\pi^S_{\min} < \pi^S_{\max}$. If $\pi^S_{\min} = \pi^S_{\max}$ instead, the previous arguments, together with Lemma 15.3 , imply that $k\left(\pi^{S'}\right) = k\left(\pi^{S''}\right)$ for any $\pi^{S'}, \pi^{S''} \in \left[\underline{\pi}^S, \pi^S_{\max}\right]$. The fact that $Q\left(\pi^S_{\max}\right) = \underline{\pi}^B$ then requires $k\left(\pi^S\right) = \underline{k}^*$ for all $\pi^S \in \left[\underline{\pi}^S, \pi^S_{\max}\right]$. It must then be the case that $\pi^S_{\max} + Q\left(\pi^S_{\max}\right) = \underline{\pi}^S + Q\left(\underline{\pi}^S\right)$, which is only possible whenever $\pi^S_{\max} = \underline{\pi}^S$—a contradiction with the assumption of the existence of a non-trivial solution to Problem (P1).

For completeness, I show that there are conditions under which $\pi^S_{\min} > \underline{\pi}^S$. Suppose that $\pi^S_{\min} = \underline{\pi}^S$. I will find conditions leading to a contradiction. Combining (P-K) and (NR-S-K), together with Proposition 15.2, imply

$$\underline{\pi}^S \geq (1-\delta)\,\pi^S_{spot}\left[l\left(\underline{\pi}^S\right)\right] + \delta\underline{\pi}^S \geq (1-\delta)\,\pi^S_{spot}\left(\phi c\underline{k}^*\right) + \delta\underline{\pi}^S = \underline{\pi}^S,$$

which in turn ensures that $l\left(\underline{\pi}^S\right) = \phi c\underline{k}^*$ and $k\left(\underline{\pi}^S\right) = \underline{k}^*$. Throughout the rest of the proof assume that the financing friction ϕ is small enough to guarantee that $\underline{k}^*_u < \underline{k}^*_B$, where $\underline{k}^*_u$ and $\underline{k}^*_B$ are given by (5.1) and (5.4), respectively. Proposition 5.1 then implies that $\underline{k}^* = \underline{k}^*_u$ so that, by the argument above, $k\left(\underline{\pi}^S\right) = \underline{k}^*_u$.

Next I show that the supplier's ex post non-reneging constraints are all binding which, by Lemma 15.1, implies that the buyer's ex post non-reneging constraints are all slack. Taking the expectation of (NR-S$_\theta$) with respect to θ and substituting into (P-K) yields

$$\underline{\pi}^S \geq (1-\delta)\left\{-(1-\phi)c\underline{k}^* + \beta^s \mathbb{E}\left[R\left(\underline{k}^*, \theta\right)\right]\right\} + \delta\underline{\pi}^S = \underline{\pi}^S,$$

where the inequality is strict whenever one of the (NR-S$_\theta$) constraint holds with strict equality, thus proving the claim.

Finally, consider the following perturbation: increase $k\left(\underline{\pi}^S\right)$ by a small $\Delta k > 0$, increase $l\left(\underline{\pi}^S\right)$ by $\Delta l = \phi c \Delta k$, and increase $T\left(\theta; \underline{\pi}^S\right)$ by $\Delta T(\theta) = \beta^s R_k\left(\underline{k}^*, \theta\right)\Delta k + (1-\delta)\phi c \Delta k$ for all $\theta \in \Theta$. The perturbation leaves constraints (NR-S-K), (F-C), and (C-C$_\theta$) unchanged and relaxes constraints (NR-S$_\theta$) and (L-C$_\theta$). Only constraints (NR-B$_\theta$) are tightened. As showed above, however, they are all slack so that the changes are valid for small enough Δk. After some calculations, it is easy to verify that the buyer's and the supplier's new profits are given by

$$\pi_\Delta^B = Q\left(\underline{\pi}^S\right) + \Delta k \underbrace{\left\{-\phi c + (1-\beta^s)\mathbb{E}\left[R_k\left(\underline{k}^*, \theta\right)\right] - (1-\delta)\phi c\right\}}_{\chi B(\phi)},$$

$$\pi_\Delta^S = \underline{\pi}^S + \Delta k \underbrace{\left\{-(1-\phi)c + \beta^s \mathbb{E}\left[R_k\left(\underline{k}^*, \theta\right)\right] + (1-\delta)\phi c\right\}}_{\chi S(\phi)},$$

respectively. Because $\partial \underline{k}_B^*/\partial\phi < 0$ and $\partial \underline{k}_u^*/\partial\phi = 0$, decreasing ϕ does not affect $\underline{k}^*$. Using the fact that $\underline{k}_u^*$ is bounded away from zero, it is then immediate that $\chi B(0) > 0$ and $\chi S(0) = 0$. Moreover, since both χB and χS are continuous in ϕ and $\partial\chi S/\partial\phi > 0$, there must exists a (strictly positive) value of ϕ such that $\pi_\Delta^B > Q\left(\underline{\pi}^S\right)$ and $\pi_\Delta^S > \underline{\pi}^S$. But then the original contract (or any other) for which the promise-keeping condition is binding cannot be the solution to Problem (P1) at $\underline{\pi}^S$, proving the result. ■

14 Proofs of Section 7

Proof of Proposition 7.1. *Part (i).* By the arguments in the text, the proposed contract maximizes the buyer's profits when the liquidity constraints (L-C$_\theta$) are ignored and the initial transfer is allowed to be negative. Because $l_u^* = 0$ and $T(\theta) \geq \beta^s R\left(k_u^*, \theta\right) \geq 0$, such constraints turn out to be slack and the solution is therefore unchanged once they are taken into consideration.

Part (ii). Let $\pi_{\max}^B$ denote the buyer's maximum level of profits under any self-enforcing relational contract and let ψ_B^* denote a self-enforcing relational contract (potentially non-stationary) generating $\pi_{\max}^B$. Recall that the solution from part (i) maximizes joint profits across all self-enforcing relational contracts. It is then immediate that $\pi_1^S\left(\psi_B^*\right) = \underline{\pi}^S(0)$, otherwise $\pi_1^S\left(\psi_B^*\right) + \pi_{\max}^B > -ck_u^* + \mathbb{E}\left[R\left(k_u^*, \theta\right)\right]$— a contradiction. Combining the promise-keeping condition (P-K) and the credibility constraints (C-C$_\theta$) implies that for all $t \geq 1$ and all $h^t \in H^t$, $\pi_t^S\left(\psi_B^* | h^t\right) = \underline{\pi}^S(0)$. The result then follows directly from constraint (NR-S-K). ■

In what follows, let $Q'_{-}\left(\pi^S\right)$ denote the left derivative of Q at π^S. The next result will be repeatedly used through the text.

Lemma 14.1 *Let ψ denote an efficient self-enforcing contract giving profits of $\pi^B(\psi)$ and $\pi^S(\psi)$ to the buyer and the supplier, respectively. If $Q'_{-}\left[\pi^S(\psi)\right] = -1$, then there is another self-enforcing contract achieving the same level of profits for both firms in which $\pi^S(\theta) \leq \pi^S$ for all $\theta \in \Theta$.*

Proof of Lemma 14.1. Let ψ satisfy the conditions of the Lemma and suppose that there is a demand realization, say θ', for which $\pi^S(\theta') > \pi^S$. Consider increasing $T(\theta')$ by $\Delta T(\theta') = \delta\left[\pi^S(\theta') - \pi^S\right]/(1-\delta)$ and decreasing $\pi^S(\theta')$ by $\Delta\pi^S(\theta') = \left[\pi^S(\theta') - \pi^S\right]$. It is easy to verify that the liquidity constraint (L-C$_\theta$) is relaxed while the supplier's profits, (NR-S-K), (NR-S$_\theta$), and (F-C) are not affected. Moreover, since $\pi^S(\theta') - \Delta\pi^S(\theta') = \pi^S \in \left[\underline{\pi}^S, \pi^S_{\max}\right]$, the credibility conditions is also satisfied. Turning to the buyer's profits,

$$\begin{aligned}
\Delta\pi^B &= -(1-\delta) f_\theta \Delta T(\theta) + \delta f_\theta \left\{Q\left[\pi^S(\theta) - \Delta\pi^S(\theta)\right] - Q\left[\pi^S(\theta)\right]\right\} \\
&= -(1-\delta) f_\theta \Delta T(\theta) - \Delta\pi^S(\theta)\, \delta f_\theta Q'_{-}\left[\pi^S(\theta)\right] \\
&= -(1-\delta) f_\theta \Delta T(\theta) + \Delta\pi^S(\theta)\, \delta f_\theta \\
&= 0,
\end{aligned}$$

where I have used the fact that $Q'_{-}\left[\pi^S(\psi)\right] = -1$ for all $\pi^S \in \left(\pi^S(\psi), \pi^S_{\max}\right]$. Finally, by some algebra it is possible to show that the change on the right-hand side of constraint (NR-B$_\theta$) equals $\Delta\pi^B = 0$. Thus the buyer's incentives to honor the ex post transfer continue to hold. Using the previous method, it is therefore possible to construct a new valid contract in which $\pi^S(\theta) \leq \pi^S$ for all $\theta \in \Theta$, as desired. ■

Proof of Theorem 7.2. *Part (i).* Step 1: The slope of Q is decreasing and weakly larger than -1. That the slope of Q is larger than minus one holds since, by increasing the initial payment, the buyer can always transfer profits to the supplier at a rate of one-to-one. The buyer may, however, do better by changing the level of capacity, the ex post transfers, or the continuation profits. That the slope of Q is decreasing follows from the concavity of Q, which in turn is implied by the possibility of randomization.

Step 2: The Pareto frontier has slope -1. Let Ψ^o denote the set of self-enforcing contracts which maximize the sum of expected discounted joint profits, i.e.

$$\Psi^o := \arg\max_{\psi\in\Psi} \; \pi^B(\psi) + \pi^S(\psi), \tag{14.1}$$

and let π^o_S denote the minimum level of profits to the supplier achieved by any contract in Ψ^o , i.e. $\pi^o_S := \arg\min_{\psi\in\Psi^o} \pi^S(\psi)$.[15] Note that by step 1 and the definition of π^o_S, for any $\pi^S \geq \pi^o_S$, it must

[15]That both Ψ^o and π^o_S are well-defined follows from the fact that the set of equilibrium payoffs is compact

be that $Q\left(\pi^S\right) = Q\left(\pi_S^o\right) - \left(\pi^S - \pi_S^o\right)$ so that the slope of Q equals -1. I will prove that $\pi_S^o = \underline{\pi}^S$. Suppose to the contrary that $\pi_S^o > \underline{\pi}^S$ and consider the following cases.

Case 1: The contract generating π_S^o is such that there is a θ, say θ', for which $T\left(\theta'\right) + l > 0$ and constraint (NR-S$_\theta$) is slack. There are two relevant subcases depending on constraint (NR-S-K). Suppose first that (NR-S-K) is slack. Note that decreasing $T\left(\theta'\right)$ by a small $\Delta T\left(\theta'\right) > 0$ only affects constraints (L-C$_\theta$) and (NR-S$_\theta$), but both are slack by assumption. The new contract is then valid and achieves the profile of profits

$$\left(\pi_S^o - (1-\delta) f_\theta \Delta T\left(\theta'\right), Q\left(\pi_S^o\right) + (1-\delta) f_\theta \Delta T\left(\theta'\right)\right).$$

Furthermore, because expected joint profits have not changed, the new contract must belong to Ψ^o. But since the supplier's profits have decreased, the previous claim contradicts the definition of π_S^o. No suppose that (NR-S-K) is binding so that $\pi_S^o = (1-\delta) l + \underline{\pi}^S$. Note that since, by assumption, $\pi_S^o > \underline{\pi}^S$, the initial transfer must be strictly positive, i.e., $l > 0$. In addition, its is without loss of generality to suppose that for all $\theta \in \Theta$, constraint (L-C$_\theta$) is slack. To see why, imagine that there is a subset of demand realizations, say $\widetilde{\Theta} \subset \Theta$, such that constraint (L-C$_\theta$) is binding for all $\theta \in \widetilde{\Theta}$. Consider decreasing $T\left(\theta'\right)$ by a small $\Delta T\left(\theta'\right) > 0$ and for all $\theta \in \widetilde{\Theta}$, increasing $T\left(\theta\right)$ by $\Delta T\left(\theta'\right) / \sum_{\widetilde{\Theta}} f_\theta$. It is easy to verify that the firms' profits remain unchanged and that constraint (NR-S-K) is satisfied (because l did not change). Moreover, since (L-C$_\theta$) is slack for θ' by assumption, all liquidity constraints are also satisfied. It only remains to check constraints (NR-B$_\theta$) and (NR-S$_\theta$). They clearly hold for $\theta \in \Theta \setminus \left(\widetilde{\Theta} \cup \{\theta'\}\right)$ since no change has been made. The decrease in $T\left(\theta'\right)$ is valid since (NR-S$_\theta$) is slack by assumption while (NR-B$_\theta$) has been relaxed. Lastly, the changes on transfers for $\theta \in \widetilde{\Theta}$ are valid since (NR-S$_\theta$) is relaxed while constraint (NR-B$_\theta$) is originally slack—since (L-C$_\theta$) is binding. Thus it is possible to assume that constraint (L-C$_\theta$) is slack for all θ, as desired. To complete the proof, consider decreasing l by a small $\Delta l > 0$. It is possible to verify that the new contract is valid and achieves profits of $Q\left(\pi_S^o\right) + \Delta l/(1-\delta)$ and $\pi_S^o - \Delta l/(1-\delta)$ to the buyer and the seller, respectively, contradiction the definition of π_S^o.

Case 2: The contract generating π_S^o is such that for all $\theta \in \Theta$, either $T\left(\theta\right) + l = 0$ or constraint (NR-S$_\theta$) is binding. It is straightforward to verify that if $T\left(\theta\right) + l = 0$, then constraint (NR-B$_\theta$) is slack. Moreover, by Lemma (15.1) below, the same holds when constraint (NR-S$_\theta$) is binding. Thus (NR-B$_\theta$) is slack for all $\theta \in \Theta$. Now, if $l > 0$, consider decreasing l by a small enough $\Delta l > 0$ and increasing each $T(\theta)$ by the same amount. Easy calculations show that the new contract does not change the firms profits, relaxes constraints (NR-S-K) and (NR-S$_\theta$), and does affect constraints (L-C$_\theta$). Only the buyer's ex post non-reneging constraints are tightened, but as mentioned earlier, they are all slack. Thus the new contract is valid and achieves the same level of profits to both firms. Furthermore, since transfers have been raised, the change ensures the existence of a demand

(which holds by standard arguments).

realization satisfying the conditions of case 1, leading to the same contradiction. Finally, suppose that $l = 0$. By Lemma 14.1, it is without loss of generality to assume that $\pi^S(\theta) \leq \pi^S$ for all θ. The supplier's profits are then given by

$$\begin{aligned}\pi^S &= (1-\delta)\left\{-ck + \mathbb{E}\left[T(\theta)\right]\right\} + \delta\mathbb{E}\left[\pi^S(\theta)\right] \\ &< (1-\delta)\left\{-ck + \beta^s\mathbb{E}\left[R(k,\theta)\right]\right\} + \delta\pi^S,\end{aligned}$$

where the second inequality follows from the previous discussion and the fact that $T(\theta) < \beta^s R(k,\theta)$ (which holds by the assumption that either $T(\theta) + l = 0$ or constraint(NR-S$_\theta$) is binding). But then

$$\pi^S < -ck + \beta^s\mathbb{E}\left[R(k,\theta)\right] < \underline{\pi}^S,$$

where the first inequality follows from the previous expression and the second follows since $\underline{\pi}^S = -ck_u^* + \beta^s E\left[R(k_u^*,\theta)\right]$, $k \geq k_u^*$ and $-ck' + \beta^s E\left[R(k',\theta)\right]$ is decreasing in k' for $k' \geq k_u^*$. Hence a contradiction.

Part (ii). By part (i), the minimum value of π^S on the Pareto frontier coincides with $\underline{\pi}^S$, i.e., $\pi^S_{\min} = \underline{\pi}^S$. Furthermore, by the credibility constraint (C-C$_\theta$) and Lemma 14.1, it follows that the profile of payoffs $\left(\underline{\pi}^S, Q\left(\underline{\pi}^S\right)\right)$ can be sustained through a stationary equilibrium, which in turn implies that the contract described in Proposition 7.1 is efficient. Now consider the following contract:

$$k\left(\pi^S\right) = k_u^*,\ l\left(\pi^S\right) = \frac{\Delta\pi^S}{1-\delta},\ \text{and } T(\theta)\left(\pi^S\right) = T_u^*(\theta) - \frac{\delta}{1-\delta}\Delta\pi^S,\ \forall\theta\in\Theta,$$

where $\Delta\pi^S := \pi^S - \underline{\pi}^S(0)$. It is immediate to verify that the proposed values of k, l, and $T(\theta)$ satisfy constraints (NR-S-K), (NR-B$_\theta$), (NR-S$_\theta$), (F-C), and (C-C$_\theta$). Moreover, the supplier's and the buyer's profits are π^S and $Q\left(\underline{\pi}^S\right) - \left(\pi^S - \underline{\pi}^S\right)$, respectively. Therefore, the new contract achieves payoffs on the Pareto frontier. It only remains to prove that the supplier's liquidity constraint is satisfied. Define $h\left(\pi^S\right)$ by $h\left(\pi^S\right) := T(\theta)\left(\pi^S\right) + l\left(\pi^S\right) - \phi ck\left(\pi^S\right) \geq 0$. Given the proposed changes, it is easy to confirm that $\partial h\left(\pi^S\right)/\partial\pi^S = 1$ and thus that the supplier's liquidity constraint relaxes as π^S increases. Thus, the proposed stationary contract is valid and achieves profits $\left(\pi^S, Q\left(\pi^S\right)\right)$, proving the result. ■

15 Proofs of Section 8

Proof of Lemma 8.1. The concavity of $Q(\pi^S)$ follows from the possibility of randomization. That $Q\left(\pi^S\right)$ is strictly decreasing holds since any Pareto efficient self-enforcing contract is sequentially Pareto efficient, i.e., all continuation profits required to generate an equilibrium in the Pareto frontier belong themselves to the Pareto frontier. Finally, to prove that the slope of $Q(\pi^S)$ is larger than minus one, consider the following two cases depending on the size of the initial transfer. When, on the one hand, $l \geq \phi c \underline{k}_u^*$, it is easy to verify that by increasing the initial payment, the buyer can transfer profits to the supplier at a rate of one-to-one without violating any of the constraints. The buyer may, however, do better by changing the level of capacity, the ex post transfers, or the continuation profits. When, on the one hand, $l < \phi c \underline{k}_u^*$, just increasing the initial payment would violate the (NR-S-K) constraint. Consider instead a small increase in l of $\Delta l > 0$, an increase in k of $\Delta k = \Delta l / \phi c$, and an increase in $T(\theta)$ of $\Delta T(\theta) = \Delta k \beta^s R_k(k, \theta)$ for all $\theta \in \Theta$. It is immediate to verify that the changes relax constraint (L-C$_\theta$) and do not affect constraint (F-C). Moreover, since continuation utilities remain constant, (NR-B$_\theta$) and (NR-S$_\theta$) are satisfied provided that

$$\Delta k R_k(k, \theta) - \Delta T(\theta) \geq \Delta k (1 - \beta^s) R_k(k, \theta) \text{ and } \Delta T(\theta) \geq \Delta k \beta^s R_k(k, \theta).$$

Substituting the proposed changes, it is straightforward to check that the previous conditions are indeed met as equalities. Lastly, since $l_t < \phi c \underline{k}_u^*$, it follows from the definition of $\underline{k}_u^*$ that, in case of a deviation, any increase in the initial payment would be used by the supplier to build more capacity. Hence, constraint (NR-S-K) is trivially satisfied and the new contract is valid. After some calculations, the changes in firms' profits are:

$$\Delta \pi^B \approx \Delta l (1 - \delta) (-\phi c + (1 - \beta^s) \mathbb{E}[R_k(k, \theta)]) / (\phi c), \tag{15.1}$$

$$\Delta \pi^S \approx \Delta l (1 - \delta) (-(1 - \phi) c + \beta^s \mathbb{E}[R_k(k, \theta)]) / (\phi c). \tag{15.2}$$

Since, by assumption, $l < \phi c \underline{k}_u^*$, constraint (F-C) implies that k must be strictly less that $\underline{k}_u^*$. The definition of $\underline{k}_u^*$ then implies that $-c + \beta^s \mathbb{E}[R_k(k, \theta)] > 0$, which in turn guarantees $\Delta \pi^S > 0$. To complete the proof, it only remains to show that $\left(\Delta \pi^B / \Delta \pi^S\right) > -1$. Using (15.1) and (15.2), the previous condition simplifies to $-c + E[R_k(k, \theta)] > 0$, which clearly holds by the definition of k^{FB} and the fact that $k < \underline{k}_u^* < k^{FB}$. ■

The next result will be repeatedly used through the text.

Lemma 15.1 *Suppose that there exists a non-trivial self-enforcing relational contract, i.e., there is a $\psi \in \Psi$ such that $\pi^B(\psi) \geq \underline{\pi}^B$ and $\pi^S(\psi) \geq \underline{\pi}^S$, with at least one strict inequality. In any Pareto efficient relational contract, any period t, and any positive probability history $h^t \in \mathcal{H}^t$, if (NR-B$_\theta$) is binding, then (NR-S$_\theta$) is slack.*

Proof of Lemma 15.1. Combining constraints (NR-B$_\theta$) and (NR-S$_\theta$) yields $\pi^S(\theta) + Q\left[\pi^S(\theta)\right] \geq \underline{\pi}^S + \underline{\pi}^B$, where the inequality becomes an equality if both constraints are binding. But in that case $\pi^S(\theta) = \underline{\pi}^S$ and $Q\left[\pi^S(\theta)\right] = \underline{\pi}^B$. It is then straightforward to check that substituting $\left(Q\left[\pi^S(\theta)\right], \pi^S(\theta)\right)$ by $\left(\pi^B(\psi), \pi^S(\psi)\right)$ for all θ does not violate any constraint and induces a Pareto improvement, contradicting the optimality of the original contract. ■

A natural question is how actions at a particular date, specially the level of investment, relate to those under spot transactions. The following result gives an answer.

Proposition 15.2 *In any efficient self-enforcing relational contract, after any period $t \geq 1$ and any positive probability history $h^t \in \mathcal{H}^t$, $k_t \geq \underline{k}^*$.*

Proof. Suppose by contradiction that $k_t < \underline{k}^*$ for some period t and some history h^t. The result is trivially true when $\underline{k}^* = 0$ so suppose that $\underline{k}^* > 0$. Consider a small increase in l of $\Delta l > 0$, an increase in k of $\Delta k = \Delta l / \phi c$, and an increase in $T(\theta)$ of $\Delta T(\theta) = \Delta k \beta^s R_k(k, \theta)$ for all $\theta \in \Theta$. These are the same changes as in the last part of Lemma (8.1) and the steps used there guarantee that constraints (L-C$_\theta$), (F-C), (NR-B$_\theta$), and (NR-S$_\theta$)) continue to hold. Before turning to constraint (NR-S-K), note that since $0 < k < \underline{k}^*$ by assumption and $\underline{k}^* = \min(\underline{k}_u^*, \underline{k}_B^*)$, the definitions of $\underline{k}_u^*$ and $\underline{k}_B^*$ imply, respectively, that

$$-c + \beta^s \mathbb{E}[R_k(k, \theta)] > 0 \text{ and } -\phi c + (1 - \beta^s)\mathbb{E}[R_k(k, \theta)] > 0. \tag{15.3}$$

To show that constraint (NR-S-K) is also satisfied, consider the following two cases depending on the size of the initial transfer. If $l_t < \phi c \underline{k}_u^*$, on the one hand, the same steps as before imply the result. If $l_t \geq \phi c \underline{k}_u^*$, on the other hand, the change in deviation profits coincides with the increase in l and the constraint holds provided that $1 - \phi^{-1} + \beta^s (\phi c)^{-1} E[R_k(k_t, \theta)] \geq 1$ or equivalently, $-c + \beta^s E[R_k(k_t, \theta)] \geq 0$. The result then follows from (15.3).

Finally, from the proof of Lemma (8.1), the changes in firms' profits are given by (15.1) and (15.2). That both firms are strictly better off after the changes, i.e., $\Delta \pi^B, \Delta \pi^S > 0$, then follows directly from (15.3). To summarize, the proposed changes meet all the constraints and lead to a Pareto improvement, contradicting the optimality of the original contract and proving the result. ■

The following Lemma provides a useful implication of local linearity of the value function.

Lemma 15.3 *Let $\underline{\pi}^S \leq \pi_*^S < \pi_{**}^S \leq \widehat{\pi}^S$. If Q is linear on the interval $[\pi_*^S, \pi_{**}^S]$, then any efficient contract must feature the same level of investment on $[\pi_*^S, \pi_{**}^S]$, i.e., $k(\pi^S) = k\left(\pi^{S'}\right)$ for all $\pi^S, \pi^{S'} \in [\pi_*^S, \pi_{**}^S]$.*

Proof. Let $\pi^{S'}$ and $\pi^{S''}$ denote two arbitrary elements of $[\pi_*^S, \pi_{**}^S]$ and let $\left\{k', l, T'(\theta), \pi^{S'}(\theta)\right\}$ and $\left\{k'', l'', T''(\theta), \pi^{S''}(\theta)\right\}$ denote the efficient contracts associated with $\pi^{S'}$ and $\pi^{S''}$, respectively. Now suppose by contradiction that $k' \neq k''$ and consider the following values: $k^\alpha = \alpha k' + (1-\alpha) k''$, $l^\alpha = \alpha l' + (1-\alpha) l''$, $T^\alpha(\theta) = \alpha T'(\theta) + (1-\alpha) T''(\theta) + \beta^s [R(k^\alpha, \theta) - R^\alpha(k, \theta)]$, and $\pi^{S\alpha}(\theta) = \alpha \pi^{S'}(\theta) + (1-\alpha)\pi^{S''}(\theta)$, where $R^\alpha(k, \theta) = \alpha R(k', \theta) + (1-\alpha) R(k'', \theta)$ and $\alpha \in (0,1)$. Using the fact that $R(k^\alpha, \theta) - R^\alpha(k, \theta) > 0$ (where the inequality holds since R is strictly concave and $k' \neq k''$), it is straightforward to verify that the new contract satisfies constraints (F-C), (L-C$_\theta$), and (C-C$_\theta$). To check (NR-B$_\theta$), observe that

$$
\begin{aligned}
&(1-\delta)\left[\beta^{s} R\left(k^{\alpha}, \theta\right)-T^{\alpha}(\theta)\right]+\delta Q\left[\pi^{S^{\alpha}}(\theta)\right] \\
&=(1-\delta)\left[\beta^{s} R^{\alpha}(k, \theta)-\alpha T^{\prime}(\theta)-(1-\alpha) T^{\prime \prime}(\theta)\right]+\delta Q\left[\pi^{S^{\alpha}}(\theta)\right] \\
&=(1-\delta)\left\{\alpha\left[\beta^{s} R\left(k^{\prime}, \theta\right)-T^{\prime}(\theta)\right]+(1-\alpha)\left[\beta^{s} R\left(k^{\prime \prime}, \theta\right)-T^{\prime \prime}(\theta)\right]\right\}+\delta Q\left[\pi^{S^{\alpha}}(\theta)\right] \\
&\geq \alpha\left\{(1-\delta)\left[\beta^{s} R\left(k^{\prime}, \theta\right)-T^{\prime}(\theta)\right]+\delta Q\left[\pi^{S^{\prime}}(\theta)\right]\right\} \\
&+(1-\alpha)\left\{(1-\delta)\left[\beta^{s} R\left(k^{\prime \prime}, \theta\right)-T^{\prime \prime}(\theta)\right]+\delta Q\left[\pi^{S^{\prime \prime}}(\theta)\right]\right\} \\
&\geq \alpha \delta \underline{\pi}^{B}+(1-\alpha) \delta \underline{\pi}^{B} \\
&=\delta \underline{\pi}^{B},
\end{aligned}
$$

where the the first inequality uses the concavity of the function Q and the second inequality holds since the original contracts are valid. The argument for (NR-S$_\theta$) is similar, hence the proof is omitted. To establish that the proposed values are valid, it only remains to verify (NR-S-K). Note that

$$
(1-\delta)\left\{l^{\alpha}-c k^{\alpha}+\mathbb{E}\left[T^{\alpha}(\theta)\right]\right\}+\delta \mathbb{E}\left[\pi^{S^{\alpha}}(\theta)\right]=\alpha \pi^{S^{\prime}}+(1-\alpha) \pi^{S^{\prime \prime}}+(1-\delta) \beta^{s} \mathbb{E}\left[R\left(k^{\alpha}, \theta\right)-R^{\alpha}(k, \theta)\right],
$$

and, after some tedious calculations,

$$
(1-\delta) \pi_{spot}^{S}\left(l^{\alpha}\right)+\delta \underline{\pi}^{S}=\alpha \pi_{d}^{S}\left(\pi^{S^{\prime}}\right)+(1-\alpha) \pi_{d}^{S}\left(\pi^{S^{\prime \prime}}\right)+(1-\delta)\left[\pi_{spot}^{S}\left(l^{\alpha}\right)-\pi_{spot}^{S \alpha}\right],
$$

where $\pi_{d}^{S}\left(\pi^{S}\right):=(1-\delta) \pi_{spot}^{S}\left[l\left(\pi^{S}\right)\right]+\delta \underline{\pi}^{S}$ and $\pi_{spot}^{S}\langle\alpha\rangle:=\alpha \pi_{spot}^{S}\left(l^{\prime}\right)+(1-\alpha) \pi_{spot}^{S}\left(l^{\prime}\right)$. Because the original contracts are valid, it then suffices to show that

$$
\beta^{s} \mathbb{E}\underbrace{\left[R\left(k^{\alpha}, \theta\right)-R^{\alpha}(k, \theta)\right]}_{>0} \geq \pi_{spot}^{S}\left(l^{\alpha}\right)-\pi_{spot}^{S}\langle\alpha\rangle . \tag{15.4}
$$

After some calculations and using the fact that, by Lemma 8.8 below, the financing constraint (F-C) is binding at both $\pi^{S^{\prime}}$ and $\pi^{S^{\prime \prime}}$, the two element on the right-hand side of (15.4) are given by

$$
\pi_{spot}^{S}\left(l^{\alpha}\right)=\left\{\begin{array}{ll}
\phi c k^{\alpha}-c \underline{k}_{u}^{*}+\beta^{s} \mathbb{E}\left[R\left(\underline{k}_{u}^{*}, \theta\right)\right] & \text { if } l^{\alpha} \geq \phi c \underline{k}_{u}^{*}, \\
-(1-\phi) c k^{\alpha}+\beta^{s} \mathbb{E}\left[R\left(k^{\alpha}, \theta\right)\right] & \text { if } l^{\alpha}<\phi c \underline{k}_{u}^{*},
\end{array}\right. \tag{15.5}
$$

and

$$
\pi_{spot}^{S}\langle\alpha\rangle=\left\{\begin{array}{ll}
\phi c k^{\alpha}-c \underline{k}_{u}^{*}+\beta^{s} \mathbb{E}\left[R\left(\underline{k}_{u}^{*}, \theta\right)\right] & \text { if } l^{\prime}, l^{\prime \prime} \geq \phi c \underline{k}_{u}^{*}, \\
-(1-\phi) c k^{\alpha}+\beta^{s} \mathbb{E}\left[R^{\alpha}(k, \theta)\right] & \text { if } l^{\prime}, l^{\prime \prime}<\phi c \underline{k}_{u}^{*}, \\
\alpha \pi_{spot}^{S}\langle\pi^{S^{\prime}}\rangle+(1-\alpha) \pi_{spot}^{S}\langle\pi^{S^{\prime \prime}}\rangle & \text { if } l^{\prime} \geq \phi c \underline{k}_{u}^{*},\, l^{\prime \prime}<\phi c \underline{k}_{u}^{*},
\end{array}\right. \tag{15.6}
$$

where

$$\pi^S_{spot}\langle\pi^{S'}\rangle = \phi ck' - c\underline{k}^*_u + \beta^s\mathbb{E}\left[R\left(\underline{k}^*_u, \theta\right)\right] \text{ and } \pi^S_{spot}\langle\pi^{S''}\rangle = -(1-\phi)ck'' + \beta^s\mathbb{E}\left[R\left(k'', \theta\right)\right].$$

Because l^α is a convex combination of l' and l'', $\max(l', l'') \geq l^\alpha \geq \min(l', l'')$. Thus $l^\alpha \geq \phi c\underline{k}^*_u$ whenever $l', l'' \geq \phi c\underline{k}^*_u$ and $l^\alpha \leq \phi c\underline{k}^*_u$ whenever $l', l'' \leq \phi c\underline{k}^*_u$. Using (15.5) and (15.6), condition (15.4) becomes $\beta^s\mathbb{E}\left[R\left(k^\alpha, \theta\right) - R^\alpha\left(k, \theta\right)\right] \geq 0$, which is clearly satisfied. Alternatively, when, say, l' is less than $\phi c\underline{k}^*_u$ but l'' is higher, depending on α, the value of l^α may be either smaller or larger than $\phi c\underline{k}^*_u$. In this case, it turns out to be hard to determine whether condition (15.4) is met. It is possible, however, to show that it is satisfied for any $\alpha \in [\widetilde{\alpha}, 1]$, where $\widetilde{\alpha} < 1$. To see why, note that since $l' < \phi c\underline{k}^*_u$ (by assumption), regardless of the value of l'', there must exist an $\alpha \in (0, 1)$, call it $\widetilde{\alpha}$, such that $l^{\widetilde{\alpha}} = \phi c\underline{k}^*_u$. Since $l' < l''$, it follows that $\partial l^\alpha/\partial\alpha < 0$ so that $l^\alpha < \phi c\underline{k}^*_u$ for any $\alpha \in [\widetilde{\alpha}, 1]$. But then, Lemma 15.6 and the fact that, as argued before, constraints (NR-S$_\theta$) are satisfied ensure that (NR-S-K) holds provided that $l^\alpha < \phi c\underline{k}^*_u$ or $\alpha \in [\widetilde{\alpha}, 1]$.

Thus there is a $\alpha \in (0, 1)$ for which the new contract satisfies all the constraints, offers the supplier's profits of

$$\pi^S(\alpha) = \alpha\pi^{S'} + (1-\alpha)\pi^{S''} + (1-\delta)\beta^s\left[R\left(k^\alpha, \theta\right) - R^\alpha\left(k, \theta\right)\right],$$

and offers the buyer profits of

$$\begin{aligned}
\pi^B(\alpha) &= (1-\delta)\left\{-l^\alpha + \mathbb{E}\left[R\left(k^\alpha, \theta\right) - T^\alpha(\theta)\right]\right\} + \delta\mathbb{E}\left\{Q\left[\pi^{S^\alpha}(\theta)\right]\right\} \\
&\geq \alpha Q\left(\pi^{S'}\right) + (1-\alpha)Q\left(\pi^{S''}\right) + (1-\delta)(1-\beta^s)\left[R\left(k^\alpha, \theta\right) - R^\alpha\left(k, \theta\right)\right], \\
&> \alpha Q\left(\pi^{S'}\right) + (1-\alpha)Q\left(\pi^{S''}\right)
\end{aligned}$$

where the inequalities use the concavity of Q, the strict concavity of R, and the assumption that $k' \neq k''$. Hence contradicting the efficiency of the original contract. ■

Proof of Proposition 8.2. *Part (i).* To prove the result for $\widehat{\pi}^S$, let $\Psi^*\left(\pi^S\right)$ denote the set of efficient contracts in which the supplier gets discounted expected profits of at least π^S. Since the objective function of Problem (P1) is continuous and the correspondence describing the set of contracts satisfying all the constraints as a function of π^S is compact-valued and continuous, the Theorem of the Maximum ensures that $\Psi^*\left(\pi^S\right)$ is non-empty, compact-valued, and upper hemi-continuous in π^S; see Stokey, Lucas, and Prescott (1989). By Proposition 8.10 and the upper hemi-continuity of $\Psi^*(\cdot)$, it then follows that there is a stationary contract achieving $\left(\widehat{\pi}^S, Q\left(\widehat{\pi}^S\right)\right)$. Note that since social surplus only depends on the level of investment, all stationary contract implementing such profits require the same k, say $\widetilde{k}$. Now suppose that there is a non-stationary contract, denoted by ψ', generating the same profits. By Lemma 14.1 and the assumption that the contract is non-stationary, it follows that $\pi^S_2(\theta) \leq \widehat{\pi}^S$ for all θ, with at least one strict inequality.

But then

$$\widehat{\pi}^S = (1-\delta)\,\pi_1^S\left(\psi'\right) + \delta\mathbb{E}\left[\pi_2^S\left(\theta\right)\right] < (1-\delta)\,\pi_1^S\left(\psi'\right) + \delta\widehat{\pi}^S$$

or $\widehat{\pi}^S < \pi_1^S\left(\psi'\right)$, which in turn implies that $k_1\left(\psi'\right) \neq \widetilde{k}$, contradicting Lemma 15.3.

To prove the result for any $\pi^S \in \left(\widehat{\pi}^S, \pi_{\max}^S\right)$, suppose to the contrary that there is a demand realization, say θ', such that $\pi^S\left(\theta'\right) < \widehat{\pi}^S$. Since, by Lemma 8.8, Q is strictly concave on $\left[\pi_{\min}^S, \widehat{\pi}^S\right)$ and, by the definition of $\widehat{\pi}^S$, linear on $\left[\widehat{\pi}^S, \pi_{\max}^S\right]$, it follows that

$$Q'_+\left[\pi^S\left(\theta'\right)\right] > Q'_+\left(\pi^S\right), \tag{15.7}$$

where $Q'_+\left(x\right)$ stands for the right derivative of Q at x. Now consider increasing $\pi^S\left(\theta'\right)$ by a small $\varepsilon > 0$. The supplier's non-reneging constraints (NR-S-K) and (NR-S$_\theta$) are both relaxed, constraints (F-C) and (L-C$_\theta$) are not affected, and, since $\pi^S\left(\theta'\right) < \widehat{\pi}^S < \pi_{\max}^S$, the credibility constraint (C-C$_\theta$) holds for sufficiently small ε. Only constraint (NR-B$_\theta$) is tightened. By Proposition 8.9, however, constraint (L-C$_\theta$) is binding, which in turn implies that the buyer's ex post non-reneging constraint is slack. Thus the change induces a valid contract in which the supplier's profits are $\pi^S + \delta f_{\theta'}\varepsilon$ and the buyer's profits are

$$Q\left(\pi^S\right) + \delta f_{\theta'}\left\{Q\left[\pi^S\left(\theta'\right)+\varepsilon\right] - Q\left[\pi^S\left(\theta'\right)\right]\right\}.$$

Furthermore, since the new contract need not be efficient,

$$Q\left(\pi^S + \delta f_{\theta'}\varepsilon\right) \geq Q\left(\pi^S\right) + \delta f_{\theta'}\left\{Q\left[\pi^S\left(\theta'\right)+\varepsilon\right] - Q\left[\pi^S\left(\theta'\right)\right]\right\}$$

or, after rearranging term,

$$\frac{Q\left(\pi^S + \delta f_{\theta'}\varepsilon\right) - Q\left(\pi^S\right)}{\delta f_{\theta'}\varepsilon} \geq \frac{Q\left[\pi^S\left(\theta'\right)+\varepsilon\right] - Q\left[\pi^S\left(\theta'\right)\right]}{\varepsilon}.$$

Taking the limit of both sides of the above inequality as ε approaches zero then yields

$$Q'_+\left(\pi^S\right) \geq Q'_+\left[\pi^S\left(\theta'\right)\right],$$

contradicting (15.7). Finally, given the previous result, the same argument used to prove the claim for $\widehat{\pi}^S$ applies to $\pi_{\max}^S$, completing the proof.

Part (ii). For any $\pi^s > \widehat{\pi}^S$, the proof is identical to that of part (ii) of Proposition 7.2 and therefore omitted. Finally, that $\left(\widehat{\pi}^S, Q\left(\widehat{\pi}^S\right)\right)$ can be generated using a stationary contract follows from part (i) and Lemma 14.1. ■

The following observation will be repeatedly used through the text.

Lemma 15.4 *When the supplier's continuation profits are maximal, the buyer's continuation prof-*

its are at its lowest level, i.e., $Q\left(\pi^S_{\max}\right) = \underline{\pi}^B$.

Proof. Suppose to the contrary that $Q\left(\pi^S_{\max}\right) > \underline{\pi}^B$. Applying the same arguments as in the proof of part (iii) of Proposition 8.1, for any arbitrarily small $\varepsilon > 0$ one can construct a new valid contract in which the buyer's and the supplier's profits are $Q\left(\pi^S_{\max}\right) - \varepsilon$ and $\pi^{S'} \geq \pi^S_{\max} + \varepsilon$, respectively. Thus, contradicting the definition of $\pi^S_{\max}$. ■

Let $\partial Q\left(\pi^S\right)$ denote the superdifferential of function Q at π^S and let λ_1, λ_2, $f_\theta \lambda_3^\theta$, $f_\theta \lambda_4^\theta$, $(1-\delta)\lambda_5$, $(1-\delta) f_\theta \lambda_6^\theta$, and $\delta f_\theta \lambda_7^\theta$ denote the Khun-Tucker multipliers associated with constraints (P-K) to (C-C$_\theta$), respectively. The first-order conditions of Problem (P1) are given by:

$$(k): \quad \mathbb{E}\left\{R_k(k,\theta)\left[1 + \beta^s\left(\lambda_3^\theta - \lambda_4^\theta\right) - \phi c \lambda_6^\theta\right]\right\} - (\lambda_1 + \lambda_2) c - \phi c \lambda_5 = 0, \tag{15.8}$$

$$(l): \quad -1 + \lambda_1 + \lambda_2\left\{1 - \frac{\partial}{\partial l}\pi^S_{spot}(l)\right\} + \lambda_5 + \mathbb{E}\left(\lambda_6^\theta\right) = 0, \tag{15.9}$$

$$T(\theta): \quad -1 + \lambda_1 + \lambda_2 - \lambda_3^\theta + \lambda_4^\theta + \lambda_6^\theta = 0, \ \forall \theta \in \Theta, \tag{15.10}$$

$$\pi^S(\theta): \quad -\left(\lambda_1 + \lambda_2 + \lambda_4^\theta + \lambda_7^{\theta+} - \lambda_7^{\theta-}\right) / \left(1 + \lambda_3^\theta\right) \in \partial Q\left[\pi^S(\theta)\right], \ \forall \theta \in \Theta. \tag{15.11}$$

And using the Envelope Theorem,

$$-\lambda_1 \in \partial Q\left(\pi^S\right). \tag{15.12}$$

Proof of Proposition 8.3. I start by proving parts (ii) and (iii) and then, at the end of the proof, I return to part (i).

Part (ii). Using the fact that $Q'\left(\pi^S\right) = -1$ for all $\pi^S > \widehat{\pi}^S$, equation (15.10), the first-order condition with respect to $T(\theta)$, yields

$$\lambda_2 - \lambda_3^\theta + \lambda_4^\theta + \lambda_6^\theta = 0, \ \forall \theta \in \Theta.$$

By Proposition 15.1, if $\lambda_4^\theta > 0$, then $\lambda_3^\theta = 0$, contradicting the previous expression. A similar contradiction can be found if $\lambda_6^\theta > 0$, suggesting that $\lambda_4^\theta = \lambda_6^\theta = 0$ and $\lambda_2 = \lambda_3^\theta$. Moreover, by Lemma 8.1, $k^o \geq \underline{k}^*_u$ (otherwise $Q'_+\left(\widehat{\pi}^S\right) > -1$, a contradiction). Equation (15.9), the first-order condition with respect to l, together with the fact that $\frac{\partial}{\partial l}\pi^S_{spot}(l) = 1$ for $l \geq \phi c \underline{k}^*_u$, then implies that $\lambda^5 = 0$. There are two two cases to consider. If $\lambda_2 = \lambda_3^\theta$, then $k^o = k^{FB}$. Alternatively, if $\lambda_2 = \lambda_3^\theta > 0$, combining (NR-S-K) and (NR-B$_\theta$) yields

$$\pi^J(k^o) - \underline{\pi}^B(\phi) - \underline{\pi}^S(\phi) = \frac{1-\delta}{\delta}\left[(-\phi c k^0 + \pi^S_{spot}(\phi c k^o) - \beta^s \mathbb{E}\left[R(k^o,\theta)\right] + c k^o\right]. \tag{15.13}$$

Next I show that if $\widehat{\pi}^S < \pi^S_{\max}$ and $\lambda_2 = \lambda_3^\theta > 0$, then $\underline{k}^*_B \geq \underline{k}^*_u$. Suppose to the contrary that $\underline{k}^*_B < \underline{k}^*_u$. Since $\lambda_2 = \lambda_3^\theta > 0$ and $Q\left(\pi^S_{\max}\right) = \underline{\pi}^B$, it follows that $T\left(\theta; \pi^S_{\max}\right) = \beta^s R(k^o,\theta)$. But then, (F-C), the definition of $\underline{k}^*_B$, together with the fact that, by Lemma 15.2 and the assumption that $\underline{k}^*_u > \underline{k}^*_B$, $k^o > \underline{k}^*_B$, imply that $\underline{\pi}^B < -\phi c k^o + (1-\beta^s)\mathbb{E}\left[R(k^o,\theta)\right] < \underline{\pi}^B$—a contradiction.

Finally, since $\underline{k}_B^* \geq \underline{k}_u^*$, $\pi_{spot}^S(\phi c k^o) = \phi c k^o - c\underline{k}_u^* + (1-\beta^s)\mathbb{E}\left[R\left(\underline{k}_u^*, \theta\right)\right]$ and, by Proposition 5.3, $\underline{\pi}^B(\phi) + \underline{\pi}^S(\phi) = \underline{\pi}^B(0) + \underline{\pi}^S(0)$. Therefore,

$$\pi^J(k^o) - \underline{\pi}^B(0) - \underline{\pi}^S(0) = \frac{1-\delta}{\delta}\left[(-\beta^s\mathbb{E}\left[R\left(k^o, \theta\right)\right] + \underline{\pi}^S(0) + ck^o\right],$$

which coincides with (7.6).

Part (iii). Suppose that $\widehat{\pi}^S = \pi_{\max}^S$. First note that $l\left(\pi_{\max}^S\right) = \phi c k^o$ by Lemma 8.1. There are two cases to consider. If NR-S-K is binding, combining (PC-B) and (NR-S-K) yields

$$\pi^J(k^o) - \underline{\pi}^J(\phi) = \frac{1-\delta}{\delta}\left[\pi_{spot}^S(\phi c k^o) - \pi^S(k^o)\right]. \tag{15.14}$$

Alternatively, if (NR-S-K) is slack, then for every $\theta \in \Theta$, either $T\left(\theta; \pi_{\max}^S\right) = 0$ or (NR-S$_\theta$) is binding. To see why, note that if both $T\left(\theta; \pi_{\max}^S\right) > 0$ and (NR-S$_\theta$) is slack, then decreasing $T\left(\theta; \pi_{\max}^S\right)$ by ε / f_θ for a small enough $\varepsilon > 0$ and increasing l by ε satisfies all constraints and does not affect social welfare. Thus, it is a valid solution to problem 5.1. Moreover, there must be at least a value of θ for which $T\left(\theta; \pi_{\max}^S\right) > 0$ as otherwise (NR-S-K) would be violated. Finally, by part (i) of Proposition 8.5, if $T\left(\theta; \pi_{\max}^S\right) > 0$, then $T\left(\theta'; \pi_{\max}^S\right) > 0$ for all $\theta' > \theta$. Let θ^* be the minimum value of θ for which (NR-S$_\theta$) is binding. Combining (PC-B) and (NR-S$_\theta$) for which $\theta \geq \theta^*$ then yields

$$\pi^J(k^o) - \underline{\pi}^J(\phi) = \frac{1-\delta}{\delta}\left[\omega(k^o) - \pi^S(k^o)\right]. \tag{15.15}$$

The result then follows from combining (15.14) and (15.15) and observing that they must hold with weak inequalities.

Part (i). That $k^o \leq k^{FB}$ when $\widehat{\pi}^S < \pi_{\max}^S$ follows from part (ii) and Proposition 7.1. For $\widehat{\pi}^S = \pi_{\max}^S$, consider the following two cases. Suppose first that (NR-S-K) is binding at $\widehat{\pi}^S = \pi_{\max}^S$ but $k^o > k^{FB}$. Observe that for any $\pi^S < \pi_{\max}^S$, (NR-S-K) requires $k\left(\pi^S\right) < k^o$. The continuity of k (part (i) of Proposition 8.12) then implies that there is small enough $\varepsilon > 0$ such that $k\left(\pi_{\max}^S - \varepsilon\right) \in \left(k^{FB}, k^o\right)$. But then Proposition 8.10 implies $Q\left(\pi_{\max}^S - \varepsilon\right) + \pi_{\max}^S - \varepsilon > Q\left(\pi_{\max}^S\right) + \pi_{\max}^S$, contradicting the strict concavity of Q for $\pi^S < \widehat{\pi}^S = \pi_{\max}^S$ (Lemma 8.8). Finally, suppose that (NR-S-K) is slack. Recall from the previous discussion that $\lambda_4^\theta = 0$, suggesting that (15.8), the first-order condition with respect to k, can then be written as

$$(1+\lambda^B)\left\{-c + \mathbb{E}\left[R_k(k^o, \theta)\right]\right\} + (1-\phi)c\left[\lambda^B - \delta\sum \lambda_4^\theta\right] + \sum \lambda_4^\theta\left[-(1-\delta)\beta^s R_k(k^o, \theta)\right] = 0.$$

Using (15.10),

$$\lambda^B - \delta\sum\lambda_4^\theta = -\left(\sum \lambda_4^\theta + \delta\sum\lambda_4^\theta f_\theta + \sum\lambda_6^\theta\right).$$

It then follows that $-c + \mathbb{E}\left[R_k(k^o, \theta)\right] \geq 0$ or equivalently, $k^o \leq k^{FB}$. ■

Proof of Proposition 8.4. Let $\underline{\phi} \in (0,1)$ be such that $\underline{k}_B^*\left(\underline{\phi}\right) = k^{FB}$, where $\underline{k}_B^*(\phi)$ is given by

(5.4). Note that $\underline{\phi}$ exists and is unique since $\lim_{\phi\downarrow 0} \underline{k}_B^*(\phi) = \infty$ and $\partial \underline{k}_B^*(\phi)/\partial\phi < 0$, respectively. That $\underline{k}_B^*(\phi)$ is strictly decreasing in ϕ also implies that $\underline{k}_B^*(\phi) > k^{FB}$ for any $\phi < \underline{\phi}$. For the first part, I show that if $\widehat{\pi}^S = \pi^S_{\max}$, then $k^o > k_B^*$. Thus, if $\phi < \underline{\phi}$ and $\widehat{\pi}^S = \pi^S_{\max}$, it must be that $k^o > k_B^* > k^{FB}$, contradicting part (i) of Proposition 8.3. Suppose to the contrary that $\widehat{\pi}^S = \pi^S_{\max}$ but $k^o \leq k_B^*$. First note that $Q\left(\pi^S_{\max}\right) = \underline{\pi}^B$ by Lemma 15.4. Moreover, by Proposition 8.2, the profile of payoffs $\left(\pi^S_{\max}, Q\left(\pi^S_{\max}\right)\right)$ can be achieved using a stationary contract and, by Lemma 8.8 below, constraint (F-C) is binding. The buyer's profits are therefore given by

$$\begin{aligned}
\pi^B &= -\phi c k^o + \mathbb{E}\left[R\left(k^o,\theta\right) - T\left(\theta\right)\right] \\
&\geq -\phi c k^o + \left(1-\beta^s\right)\mathbb{E}\left[R\left(k^o,\theta\right)\right],
\end{aligned}$$

where the inequality follows since $\beta^s R\left(k^o,\theta\right) \geq T\left(\theta\right)$ (which holds by (NR-B$_\theta$) and the fact that $Q\left(\pi^S_{\max}\right) = \underline{\pi}^B$). But since, by assumption, the contract is non-trivial (i.e., $k^o > \underline{k}^* = \min\left\{\underline{k}_B^*, \underline{k}_u^*\right\} \geq 0$) and $k^o \leq \underline{k}_B^*$, the definition of $\underline{k}_B^*$ implies

$$\begin{aligned}
-\phi c k^o + \left(1-\beta^s\right)\mathbb{E}\left[R\left(k^o,\theta\right)\right] &> -\phi c \underline{k}^* + \left(1-\beta^s\right)\mathbb{E}\left[R\left(\underline{k}^*,\theta\right)\right] \\
&= \underline{\pi}^B,
\end{aligned}$$

contradicting Lemma 15.4. $\widehat{\pi}^S < \pi^S_{\max}$ for $\delta > \delta\left(\phi\right)$.

Now let $\delta_\phi := \inf\{\delta \in (0,1) \,|\widehat{\pi}^S < \pi^S_{\max}\}$. Because constraints (NR-S-K), (NR-B$_\theta$), and (NR-S$_\theta$) all relax as δ increases, provided that both firms obtain profits above their outside options, any investment level and any set of feasible transfers can be implemented by a sufficiently high (but less than one) discount factor. It follows that δ_ϕ is well-defined. The same logic and the fact that, by part (i) of Proposition 8.3, $k^o \leq k^{FB}$ ensure that k^o is weakly increasing in δ. Fix ϕ and suppose that given δ, the solution to problem P-SO features $\widehat{\pi}^S\left(\delta\right) < \pi^S_{\max}\left(\delta\right)$. I shall prove that $\widehat{\pi}^S\left(\delta'\right) < \pi^S_{\max}\left(\delta'\right)$ for any $\delta' > \delta$. Suppose first that $k^o\left(\delta\right) = k^{FB}$. Since $\partial k^o\left(\delta\right)/\partial\delta \geq 0$ and $k^o \leq k^{FB}$, it follows that $k^o\left(\delta'\right) = k^{FB}$ for any $\delta' > \delta$. Moreover, it should be clear that the same transfers that solve problem P-SO under δ are also a solution under δ', which in turn implies that $\widehat{\pi}^S\left(\delta'\right) \leq \widehat{\pi}^S\left(\delta\right) < \pi^S_{\max}\left(\delta\right) = \pi^S_{\max}\left(\delta'\right)$, where the last equality holds since $\pi^S_{\max} = \pi^J\left(k^o\right) - \underline{\pi}^B$ and, as argued before, $k^o\left(\delta\right) = k^o\left(\delta'\right) = k^{FB}$. Alternatively, if $k^o\left(\delta\right) < k^{FB}$, the proof of Proposition 8.3 implies that constraint (NR-S-K) is binding, i.e.,

$$\pi^J\left(k^o\left(\delta\right)\right) - \widehat{\pi}^B\left(k^o\left(\delta\right)\right) = \left(1-\delta\right)\pi^S_{spot}\left(\phi c k^o\left(\delta\right)\right) + \delta\underline{\pi}^S, \tag{15.16}$$

where I have used the fact that $\phi c k^o\left(\delta\right) = l$ at $\pi^S = \widehat{\pi}^S$ (Lemma 8.8). Now suppose that $\widehat{\pi}^S\left(\delta'\right) = \pi^S_{\max}\left(\delta'\right)$ or, using the fact that $Q\left(\pi^S_{\max}\left(\delta'\right)\right) = \underline{\pi}^B$, that $\widehat{\pi}^B\left(k^o\left(\delta'\right)\right) = \underline{\pi}^B < \widehat{\pi}^B\left(k^o\left(\delta\right)\right)$. Equation (15.16) then requires $\Delta\left\{\left(1-\delta\right)\pi^S_{spot}\left(\phi c k^o\left(\delta\right)\right)\right\} > \Delta\pi^J\left(k^o\left(\delta\right)\right)$, which in turn implies that $\Delta\pi^S_{\max} > \Delta\pi^J\left(k^o\left(\delta\right)\right)$, a contradiction. ■

The next Lemma, which was omitted from the main text, offers some extra insights into the mechanism behind the possibility of over-investment.

Lemma 15.5 *In any efficient self-enforcing contract, if there exists a positive probability history $h^t \in \mathcal{H}^t$ such that $k_t > k^{FB}$, then:*

(i) Either (F-C) is binding or there exists at least one demand realization such that (L-C$_\theta$) is binding.

(ii) For each demand realization θ, if the supplier's liquidity constraints (L-C$_\theta$) is slack, then his ex post non-reneging constraints (NR-S$_\theta$) is binding.

Proof of Lemma 15.5. *Part (i).* Suppose, by contradiction, that the claim is false and consider a small decrease in l_t of $\Delta l > 0$, a decrease in k_t of $\Delta k = \Delta l/\phi c$, and a decrease in $T_t(\theta)$ of $\Delta T(\theta) = \Delta k \beta^s R_k(k_t, \theta)$ for all $\theta \in \Theta$. Because (L-C$_\theta$) and (F-C) are slack by assumption, they are satisfied for sufficiently small Δl. It is also easy to check that constraints (NR-B$_\theta$) and (NR-S$_\theta$) are unchanged. Finally, to verify that (NR-S-K) still holds, note that since capacity is beyond its first-best level, the initial payment must be larger that $\phi c \underline{k}_u^*$. The condition then reduces to $1 - \phi^{-1} + \beta^s (\phi c)^{-1} E[R_k(k_t, \theta)] \leq 1$ which holds with strict inequality since the net marginal contribution of capacity to joint profits is negative, i.e., $-c + E[R_k(k_t, \theta)] < 0$. Thus the changes induce a valid contract. Now consider an additional change in the initial payment of $\Delta' l = \left(\Delta \pi_t^B - \xi\right)/(1-\delta)$, where $\Delta \pi_t^B$ denotes the change in the buyer's expected profits at time t induced by the first perturbation. Observe that $\Delta' l$ can be either positive or negative but, since $\Delta \pi_t^B$ can be made arbitrarily small by reducing Δl, it should be clear that there are small enough values of Δl and ξ so that both (F-C) and (L-C$_\theta$) are still satisfied (recall that, by assumption, both constraints are originally slack). By some algebra, it is possible to confirm that the total changes in the buyer's and the supplier's profits are $\Delta' \pi_t^B = \xi$ and $\Delta' \pi_t^S = \Delta \pi_t^B + \Delta \pi_t^S - \xi$, respectively. Finally, since $\Delta \pi_t^J := \Delta \pi_t^B + \Delta \pi_t^S \propto c - E[R_k(k_t, \theta)] > 0$, it follows that there exists a small enough $\xi > 0$ such that $\Delta' \pi_t^B, \Delta' \pi_t^S > 0$, a contradiction.

Part (ii). Suppose instead that there is a history h^t such that $k_t > k^{FB}$ and a demand realization $\widetilde{\theta}$ such that both (NR-S$_\theta$) and (L-C$_\theta$) are slack. The proof is divided in two cases.

Case 1: Suppose first that constraint (NR-B$_\theta$) is slack for some θ, say θ^B. Let $\Theta^B \subseteq \Theta$ denotes the set of states in which constraint (NR-B$_\theta$) holds with equality and note that if $\theta \in \Theta^B$, then both (NR-S$_\theta$) and (L-C$_\theta$) are slack: the former by Lemma 15.1 and the latter by constraints (NR-B$_\theta$) and the fact that $T_t(\theta) \geq \beta^s R(k_t, \theta)$ since $Q\left(\pi^S\right) \geq \underline{\pi}^B$ for all $\pi^S \in \left[\underline{\pi}^S, \pi^S_{\max}\right]$. Now consider the following perturbation: decrease l_t by a small $\Delta l > 0$, decrease k_t by $\Delta k = \Delta l/\phi c$, and, for $\theta \in \Theta^B$, decrease $T_t(\theta)$ by $\Delta T(\theta) = \Delta k \beta^s R(k_t, \theta)$. It is easy to verify that the changes only affect constraints (L-C$_\theta$) for $\theta \in \Theta^B$ and constraints (NR-B$_\theta$) and (NR-S$_\theta$) for $\theta \notin \Theta^B$: the first two are tightened while the last one is relaxed. As mentioned earlier, however, (L-C$_\theta$), $\theta \in \Theta^B$,and (NR-B$_\theta$), $\theta \notin \Theta^B$, are all slack and thus continue to hold for sufficiently small Δl. Before checking constraint (NR-S-K), I turn to the impact of the perturbation on the firms' profits. After some calculations, the changes in the buyer's and the supplier's period t profits are given by

$$\Delta\pi_t^B = \Delta l - \Delta k \mathbb{E}\left[R\left(k_t,\theta\right)\right] + \sum_{\theta\in\Theta^B} f_\theta \beta^s R\left(k_t,\theta\right),$$
$$\Delta\pi_t^S = -\Delta l + c\Delta k - \sum_{\theta\in\Theta^B} f_\theta \beta^s R\left(k_t,\theta\right),$$

respectively. Note that, since $k_t > k^{FB}$ and $\Delta k > 0$, the change in joint profits, i.e., $\Delta\pi_t^J := \Delta\pi_t^B + \Delta\pi_t^S = \Delta k \left\{c - \mathbb{E}\left[R\left(k_t,\theta\right)\right]\right\}$, is strictly positive. Observe also that if $\Delta\pi_t^B, \Delta\pi_t^S \geq 0$, then constraint (NR-S-K) is satisfied since the the initial payment has decreased. Thus the perturbation achieves a valid Pareto improvement, a contradiction. Now consider the case in which one of the firm is made worse off. If $\Delta\pi_t^B < 0$, then change $T_t\left(\widetilde{\theta}\right)$ by $\Delta T\left(\widetilde{\theta}\right) = \Delta\pi_t^B / f_{\widetilde{\theta}}$. Alternatively, if $\Delta\pi_t^S < 0$, then change $T_t\left(\theta^B\right)$ by $\Delta T\left(\theta^B\right) = \Delta\pi_t^B / f_{\widetilde{\theta}}$. In either case, only constraints (L-C$_\theta$) and (NR-S$_\theta$), $\theta \in \left\{\widetilde{\theta}, \theta^B\right\}$, are tightened. They are both, however, slack by assumption so that they will continue to hold for small enough Δl. The perturbation is then valid and induce a new contract in which the buyer's profits are the same as under the original agreement and the supplier's profits at time $t$change by $\Delta'\pi_t^S = \Delta\pi_t^J > 0$. Finally, (NR-S-K) holds since $\Delta'\pi_t^S > 0$ while l —and therefore the supplier's deviation payoffs—has decreased.

Case 2: Now suppose that for all $\theta \in \Theta$, constraint (NR-B$_\theta$) is binding. First note that, as mentioned in case 1, both (L-C$_\theta$) and (NR-S$_\theta$) must be slack. Now consider decreasing k_t by a small $\Delta k > 0$ and decreasing $T\left(\theta\right)$ by $\Delta T\left(\theta\right) = R_k(k_t,\theta)\Delta k$. The credibility constraints (C-C$_\theta$) are not affected and constraints (NR-B$_\theta$) and (F-C) are both relaxed. Constraints (NR-S$_\theta$) are tightened while constraints (L-C$_\theta$) can be relaxed or tightened. Either way, as mentioned earlier, they are all originally slack so that they will be satisfied for sufficiently small Δk. Furthermore, it is straightforward to verify that the buyer's profits are unchanged while the supplier's profits increase by $\Delta\pi_t^S = (1-\delta)(c - R_k(k_t,\theta))\Delta k > 0$, where the inequality follows since $k_t > k^{FB}$ by assumption. It only remains to check constraints (NR-S-K). But it is immediate that it holds since $\Delta\pi_t^S > 0$ and the supplier's deviation payments are unchanged (because $\Delta l = 0$). Thus the perturbations induce a valid Pareto improvement, contradicting the efficiency of the original contract. ■

Proof of Proposition 8.5. Incorporating the stationary of the contract, the buyer's and the supplier's ex post non-reneging constraints can be respectively written as

$$Q\left(\pi^S\right) - \underline{\pi}^B \geq \frac{1-\delta}{\delta}\left[T\left(\theta\right) - \beta^s R\left(k,\theta\right)\right], \ \forall\theta \in \Theta, \tag{15.17}$$
$$\pi^S - \underline{\pi}^S \geq -\frac{1-\delta}{\delta}\left[T\left(\theta\right) - \beta^s R\left(k,\theta\right)\right], \ \forall\theta \in \Theta. \tag{15.18}$$

Part (i). Step 1: For every stationary contract, there is an equivalent contract in which for all $\theta \in \Theta$, either $T\left(\theta\right) = 0$ or $T\left(\theta\right) = \beta^s R\left(k,\theta\right) + A$. Suppose instead that there are two realizations of market conditions, say θ' and θ'', such that $T\left(\theta'\right), T\left(\theta''\right) > 0$ but $T\left(\theta'\right) - \beta^s R\left(k,\theta'\right) > T\left(\theta''\right) -$

$\beta^s R(k,\theta'')$. Let

$$A := \frac{f_{\theta'}\left[T(\theta') - \beta^s R(k,\theta')\right] + f_{\theta''}\left[T(\theta'') - \beta^s R(k,\theta'')\right]}{f_{\theta'} + f_{\theta''}}$$

and, for $\theta \in \{\theta', \theta''\}$, consider replacing $T(\theta)$ by $\widetilde{T}(\theta) = \beta^s R(k,\theta) + A$. Because the expected value of transfers remains the same, both firms' profits are unchanged. Constraint (F-C) is not affected and, since $\Delta\pi^S, \Delta l = 0$, constraint (NR-S-K) is satisfied. Furthermore, since the left-hand sides of (15.17) and (15.18) are independent of θ and A is just a convex combination of the term in brackets in the right-hand side, constraints (NR-B$_\theta$) and (NR-S$_\theta$) continue to hold. In only remains to verify the supplier's liquidity constraints. It clearly holds for θ'' since, given the definition of A, $\widetilde{T}(\theta'') > T(\theta'')$. Because $\widetilde{T}(\theta) < T(\theta)$, however, the perturbation may violate constraint (L-C$_\theta$) for θ'. If (L-C$_\theta$), $\theta = \theta'$, is violated, then set $\widetilde{T}(\theta') = 0$ and $\widetilde{T}(\theta'') = \beta^s R(k,\theta'') + A''$, where

$$A'' := \frac{f_{\theta'} T(\theta') + f_{\theta''}\left[T(\theta'') - \beta^s R(k,\theta'')\right]}{f_{\theta''}},$$

and the change is valid by the above arguments. Alternatively, if (L-C$_\theta$), $\theta = \theta'$ is satisfied, then perform the original perturbation. Repeating the procedure above for each pair of demand realization yields the result.

Step 2: To complete the proof, I show that if $T(\theta) = 0$, then $-\beta^s R(k,\theta) \geq A$. Suppose to the contrary that there is a realization of demand conditions, say θ', for which $T(\theta') = 0$ but $-\beta^s R(k,\theta') < A$. Let θ'' denote one of the values of θ featuring positive transfers (there should be at least one of such values, otherwise the supplier's participating constraint would be violated). Note that

$$T\left(\theta''\right) - \beta^s R\left(k,\theta''\right) = A > \underbrace{T\left(\theta'\right)}_{0} - \beta^s R\left(k,\theta'\right),$$

where the inequality follows by assumption. Performing the same perturbations as in step 1 then implies the existence of an equivalent contract in which $T(\theta') > 0$. Clearly, it is possible to repeat step 1 and the previous argument until the desired result holds.

Part (ii). The proof is identical to that of Part (ii) of Proposition 7.2 and hence omitted. ■

Proof of Proposition 8.6. Part (i) follows directly from Proposition 8.3, together with the fact that k_u^*, the investment level without credit frictions, is independent of θ. For the second part, suppose that $\pi^S_{\max} = \widehat{\pi}^S$ and let λ^B, λ_2, $f_\theta\lambda_3^\theta$, $f_\theta\lambda_4^\theta$, λ_5, and $f_\theta\lambda_6^\theta$, denote the Khun-Tucker multipliers associated with the respective constraints of problem P-SO. After some calculations, the change in social welfare due to an increase in ϕ is given by

$$\Delta SW = -\Delta\underline{\pi}^B\left[\lambda^B + \delta\mathbb{E}\left(\lambda_3^\theta\right)\right] - \delta\Delta\underline{\pi}^S\left[\lambda_2 + \mathbb{E}\left(\lambda_4^\theta\right)\right] - ck^o\left[\lambda_5 + \mathbb{E}\left(\lambda_6^\theta\right)\right].$$

Suppose first that $\underline{k}_B^* > \underline{k}_u^*$ or equivalently, $\phi < \widehat{\phi}$, where $\widehat{\phi}$ was defined in Proposition 5.2. Note that the latter result also implies that $\Delta\underline{\pi}^S = -\Delta\underline{\pi}^B = c\underline{k}_u^* > 0$. Thus,

$$\Delta SW = c\underline{k}_u^* \left\{\lambda^B - \delta\left[\lambda_2 + \mathbb{E}\left(\lambda_4^\theta - \lambda_3^\theta\right)\right]\right\} - ck^o\left[\lambda_5 + \mathbb{E}\left(\lambda_6^\theta\right)\right].$$

Rearranging the first-order condition with respect to l yields

$$\lambda_5 + \mathbb{E}\left(\lambda_6^\theta\right) = \lambda^B - \delta c\left[\lambda_2 + \mathbb{E}\left(\lambda_4^\theta\right) - \mathbb{E}\left(\lambda_3^\theta\right)\right].$$

Hence,

$$\Delta SW = -c\left(k^o - \underline{k}_u^*\right)\left[\lambda_5 + \mathbb{E}\left(\lambda_6^\theta\right)\right] < 0,$$

where the last inequality holds since $k^o > \underline{k}_B^* > \underline{k}_u^*$ by Proposition 8.3 and $\lambda_5 > 0$ by Lemma 8.2.

Now suppose that $\underline{k}_B^* \leq \underline{k}_u^*$ or equivalently, $\phi \geq \widehat{\phi}$. In addition, suppose that (NR-S-K) is binding, i.e., $\lambda_2(\phi) > 0$. Using (PC-B), constraint (NR-S-K) can be written as

$$\pi^J(k^o) = (1-\delta)\pi_{spot}^S(\phi c k^o) + \delta\underline{\pi}^S(\phi) + \underline{\pi}^B(\phi).$$

By the continuity of the multipliers, $\lambda_2(\phi + \varepsilon) > 0$ for small enough $\varepsilon > 0$. Thus

$$\frac{\partial \pi^J(k^o(\phi))}{\partial k}\frac{\partial k^o(\phi)}{\partial \phi} = (1-\delta)c\frac{\partial \pi_{spot}^S(x)}{\partial x}\left[k^o(\phi) + \frac{\partial k^o(\phi)}{\partial \phi}\phi\right] + \left[\delta\frac{\partial \underline{\pi}^S(\phi)}{\partial \phi} + \frac{\partial \underline{\pi}^B(\phi)}{\partial \phi}\right]$$

or

$$\frac{\partial k^o(\phi)}{\partial \phi} = \frac{w(\phi)}{z(\phi)},$$

where

$$w(\phi) = (1-\delta)c\frac{\partial \pi_{spot}^S(x)}{\partial x}\left[k^o(\phi)\right] + \left[\delta\frac{\partial \underline{\pi}^S(\phi)}{\partial \phi} + \frac{\partial \underline{\pi}^B(\phi)}{\partial \phi}\right],$$

$$z(\phi) = \frac{\partial \pi^J(k^o(\phi))}{\partial k} - (1-\delta)\frac{\partial \pi_{spot}^S(x)}{\partial x}c\phi.$$

The existence of ϕ_2 and ϕ_3 then follows since, by the Theorem of the Maximum, k^o is continuous in ϕ, which in turn ensures that the same holds for $w(\phi)$ and $z(\phi)$. ■

Proof of Proposition 8.7. To prove the first claim, note that the firms' expected joint profits under spot transactions can be made arbitrary small by choosing appropriate values of β^s and ϕ. The result then follows directly from Theorem 7.2 and part (iii) of Proposition 8.3. To prove the second claim, first note that if $\beta^s \leq 1/2$, then $\underline{k}_B^* > \underline{k}_u^*$ for any $\phi \in [0,1)$. The result then holds because, by the first case in the proof of Proposition 8.6, $\partial\pi^J(\phi)/\partial\phi < 0$ when $\underline{k}_B^* > \underline{k}_u^*$. ■

Proof of Lemma 8.8. *Part (i).* Let $\pi^S \in \left[\pi_{\min}^S, \widehat{\pi}^S\right]$ and suppose by contradiction that $l > \phi c k$. First note that the fact that $l > \phi c k$ together with Lemma 15.2 imply that $\pi^S > \underline{\pi}^S$. The rest of the argument is divided in two cases and essentially follows the same steps as the proof of part (i) of Proposition 7.2.

Case 1: The contract generating π^S is such that there is a θ, say θ', for which $T(\theta') + l - \phi ck > 0$ and constraint (NR-S$_\theta$) is slack. There are two relevant subcases depending on constraint (NR-S-K). Suppose first that (NR-S-K) is slack. Note that decreasing $T(\theta')$ by a small $\Delta T(\theta') > 0$ only affects constraints (L-C$_\theta$) and (NR-S$_\theta$), but both are slack by assumption. The new contract is then valid and achieves the profile of profits

$$\left(\pi^S - (1-\delta) f_\theta \Delta T(\theta'), Q\left(\pi^S\right) + (1-\delta) f_\theta \Delta T(\theta')\right).$$

Using the fact that the propose contract need not be efficient, it must be that

$$Q\left(\widetilde{\pi}^S\right) \geq Q\left(\pi^S\right) + \pi^S - \widetilde{\pi}^S \text{ for any } \widetilde{\pi}^S \in \left(\pi^S - (1-\delta) f_\theta \Delta T(\theta'), \pi^S\right),$$

which contradicts the fact that $Q'_-\left(\pi^{S'}\right), Q'_+\left(\pi^{S'}\right) > -1$ for all $\pi^{S'} < \widehat{\pi}^S$. Now suppose that (NR-S-K) is binding so that $\pi^S = (1-\delta)\pi^S_{spot}(l) + \underline{\pi}^S$. First note that its is without loss of generality to suppose that for all θ, constraint (L-C$_\theta$) is slack. To see why, suppose that there is a subset of demand realizations, say $\widetilde{\Theta} \subset \Theta$, such that constraint (L-C$_\theta$) is binding for all $\theta \in \widetilde{\Theta}$. Consider decreasing $T(\theta')$ by a small $\Delta T(\theta') > 0$ and for all $\theta \in \widetilde{\Theta}$, increasing $T(\theta)$ by $\Delta T(\theta') / \sum_{\widetilde{\Theta}} f_\theta$. It is easy to verify that the firms' profits remain unchanged and that constraint (NR-S-K) is satisfied (because l did not change). Since, by assumption, (L-C$_\theta$) is slack for θ', all liquidity constraints are also satisfied. It only remain to check constraints (NR-B$_\theta$) and (NR-S$_\theta$). They clearly hold for $\theta \in \Theta \setminus \left(\widetilde{\Theta} \cup \{\theta'\}\right)$ since no change has been made. The decrease in $T(\theta')$ is valid since (NR-S$_\theta$) is slack by assumption while (NR-B$_\theta$) has been relaxed. Lastly, the changes on transfers for $\theta \in \widetilde{\Theta}$ are valid since (NR-S$_\theta$) is relaxed while constraint (NR-B$_\theta$) is originally slack—since (L-C$_\theta$) is binding. Thus it is possible to assume that constraint (L-C$_\theta$) is slack for all θ. To complete the proof, consider decreasing l by a small $\Delta l > 0$. It is possible to verify that the new contract is valid and achieves profits of $Q(\pi_S) + \Delta l/(1-\delta)$ and $\pi_S - \Delta l/(1-\delta)$ to the buyer and the seller, respectively, leading to the same contradiction as in the first subcase.

Case 2: The contract generating π^S is such that for all $\theta \in \Theta$, either $T(\theta) + l - \phi ck = 0$ or constraint (NR-S$_\theta$) is binding. It is straightforward to verify that if $T(\theta) + l = 0$, then constraint (NR-B$_\theta$) is slack. Moreover, by Lemma (15.1), the same holds when constraint (NR-S$_\theta$) is binding. Thus (NR-B$_\theta$) is slack for all θ. Now consider decreasing l by a small $\Delta l > 0$ and increasing each $T(\theta)$ by the same amount. Easy calculations show that the new contract does not change the firms profits, relaxes constraints (NR-S-K) and (NR-S$_\theta$), and does affect constraints (L-C$_\theta$). Only the buyer's ex post non-reneging constraints (NR-B$_\theta$) and the supplier's financing constraint (F-C) are tightened, but as mentioned earlier, they are all slack and will continue to hold for sufficiently small Δl. Thus the new contract is valid and achieves the same level of profits to both firms. Furthermore, since transfers have been raised, the change ensures the existence of a demand realization satisfying the conditions of case 1, leading to the same contradiction.

Part (ii). Suppose to the contrary that Q is linear on some interval $\left[\pi^{S'},\pi^{S''}\right]$, where $\pi^{S'}\geq\pi^{S}_{\min}$ and $\pi^{S''}<\widehat{\pi}^{S}$. Without loss of generality, assume that the interval is as large as possible, i.e.,

$$\pi^{S'}=\inf\left\{\pi^{S}|Q'_{+}\left(\pi^{S}\right)=Q'\left(\pi^{S}\right),\pi^{S}\in\left(\pi^{S'},\pi^{S''}\right)\right\},\tag{15.19}$$

$$\pi^{S^{\prime\prime}}=\sup\left\{\pi^{S}|Q'_{-}\left(\pi^{S}\right)=Q'\left(\pi^{S}\right),\pi^{S}\in\left(\pi^{S'},\pi^{S''}\right)\right\}.\tag{15.20}$$

Step 1: Since, by Lemma 15.3, the level of investment is constant for all $\pi^{S}\in\left[\pi^{S'},\pi^{S''}\right]$ and, by Lemma 8.8, constraint (F-C) is binding, the initial transfer l and therefore $\pi^{S}_{aut}(l)$ must also remain constant. It is then easy to verify that (NR-S-K) is slack for all $\pi^{S}\in\left[\pi^{S'},\pi^{S''}\right]$, i.e., $\lambda_2=0$. Combining equations (15.8) and (15.10), and using the fact that $\lambda_2=0$ yield

$$-\phi c+\mathbb{E}\left[R_k\left(k,\theta\right)+\beta^{s}\lambda_6^{\theta}\right]-\lambda_1\left(1-\phi\right)c=0,$$

which, using the implicit function theorem, implies that $\lambda_6^{\theta}\left(\pi^{S}\right)$ is constant for all $\pi^{S}\in\left[\pi^{S'},\pi^{S''}\right]$. In addition, (15.10) and the fact that λ_6^{θ} is constant imply that λ_4^{θ} is also constant. Now consider $\pi^{S'}+\varepsilon$ and $\pi^{S''}$ for a small $\varepsilon>0$. If $\lambda_4^{\theta}>0$, then

$$\begin{aligned}(1-\delta)\,T\left(\theta;\pi^{S'}+\varepsilon\right)+\delta\pi^{S}\left(\theta;\pi^{S'}+\varepsilon\right)&=(1-\delta)\,\beta^{s}R\left(k,\theta\right)+\delta\underline{\pi}^{S}\\&=(1-\delta)\,T\left(\theta;\pi^{S''}\right)+\delta\pi^{S}\left(\theta;\pi^{S''}\right).\end{aligned}$$

Alternatively, when $\lambda_4^{\theta}=0$, (15.10) requires $\lambda_6^{\theta}>0$ so that $T\left(\theta;\pi^{S'}+\varepsilon\right)=T\left(\theta;\pi^{S''}\right)=0$ by Proposition (8.8). Furthermore, it is easy to verify that $\lambda_6^{\theta}>0$ implies $\lambda_3^{\theta}=0$. Hence, by equations (15.11), (15.19), and (15.20),

$$\lambda_4^{\theta}=0\implies\pi^{S}\left(\theta;\pi^{S'}+\varepsilon\right),\pi^{S}\left(\theta;\pi^{S''}\right)\in\left[\pi^{S'},\pi^{S''}\right].\tag{15.21}$$

Step 2: Contradiction. Using the previous results, the promise-keeping constraints associated with $\pi^{S'}+\varepsilon$ and $\pi^{S''}$ can be, respectively, written as:

$$-\left(1-\delta\right)\left(1-\phi\right)ck+\sum_{\Theta^1}f_\theta\left[(1-\delta)\,\beta^{s}R\left(k,\theta\right)+\delta\underline{\pi}^{S}\right]+\delta\sum_{\Theta^2}f_\theta\pi^{S}\left(\theta;\pi^{S'}+\varepsilon\right)=\pi^{S'}+\varepsilon,$$

$$-\left(1-\delta\right)\left(1-\phi\right)ck+\sum_{\Theta^1}f_\theta\left[(1-\delta)\,\beta^{s}R\left(k,\theta\right)+\delta\underline{\pi}^{S}\right]+\delta\sum_{\Theta^2}f_\theta\pi^{S}\left(\theta;\pi^{S''}\right)=\pi^{S''},$$

where $\Theta^1,\Theta^2\subseteq\Theta$ denote the set of states in which $\lambda_4^{\theta}>0$ and $\lambda_4^{\theta}=0$, respectively. Combining the last two equations yields

$$\delta\sum_{\Theta^2}f_\theta\left[\pi^{S}\left(\theta;\pi^{S''}\right)-\pi^{S}\left(\theta;\pi^{S'}+\varepsilon\right)\right]=\pi^{S''}-\left(\pi^{S'}+\varepsilon\right).\tag{15.22}$$

But, by (15.21),

$$\delta \sum_{\Theta^2} f_\theta \left[\pi^S\left(\theta; \pi^{S''}\right) - \pi^S\left(\theta; \pi^{S'} + \varepsilon\right)\right] \leq \delta\left[\pi^{S''} - \pi^{S'}\right] \sum_{\Theta^2} f_\theta$$
$$< \left[\pi^{S''} - \pi^{S'}\right],$$

which, for ε sufficiently enough, contradicts (15.22). ■

Proof of Proposition 8.9. Suppose that $T(\theta) + l - \phi ck > 0$ but $\pi^S(\theta) < \widehat{\pi}^S$. Consider the following changes to the contract associated with π^S : increase $\pi^S(\theta)$ by a small $\Delta > 0$ and decrease $T(\theta)$ by $\delta\Delta/(1-\delta)$. It is easy to verify that the new contract satisfies all the constraints, does not change the supplier's profits, and increases the buyer's profits by $\delta\Delta f_\theta\left\{1 + Q'_+\left[\pi^S(\theta)\right]\right\} > 0$, where the inequality follows from the concavity of Q, the definition of $\widehat{\pi}^S$, and the fact that $\pi^S(\theta) < \widehat{\pi}^S$. But this contradicts the optimality of the original contract. ■

Lemma 15.6 *Suppose $\pi^S < \widehat{\pi}^S$ and $k < k_u^*$, then constraint (NR-S-K) is slack, i.e., $\lambda_2 = 0$.*

Proof. Since $\pi^S < \widehat{\pi}^S$, Lemma 8.8 implies $l = \phi ck$. In addition, since $k < k_u^*$, it follows that $l < \phi ck_u^*$. Thus $\pi^S_{aut}(l) = -(1-\phi)ck + \beta^s E\left[R(k,\theta)\right]$ and constraint (NR-S-K) can be written as

$$(1-\delta)\left\{-(1-\phi)ck + \mathbb{E}\left[T(\theta)\right]\right\} + \delta\mathbb{E}\left[\pi^S(\theta)\right] \geq (1-\delta)\left\{-(1-\phi)ck + \beta^s\mathbb{E}\left[R(k,\theta)\right]\right\} + \delta\underline{\pi}^S$$

or equivalently,

$$(1-\delta)\mathbb{E}\left[T(\theta)\right] + \delta\mathbb{E}\left[\pi^S(\theta)\right] \geq (1-\delta)\beta^s\mathbb{E}\left[R(k,\theta)\right] + \delta\underline{\pi}^S.$$

It is straightforward to verify that the previous expression is implied by the supplier's ex post non reneging constraints, i.e., (NR-S$_\theta$), $\theta \in \Theta$, which guarantees the desired result. ■

Proof of Proposition 8.10. Suppose by contradiction that $\pi^S < \widehat{\pi}^S$ but $\pi^S(\theta) < \pi^S$ for some θ. First note that since $\pi^S(\theta) < \pi^S < \widehat{\pi}^S \leq \pi^S_{\max}$, $\lambda_7^{\theta+} = 0$. In addition, $T(\theta') = 0$ by Lemma 8.8 and Proposition 8.9. It is easy to see then that (NR-B$_\theta$) is slack (i.e., $\lambda_3^\theta = 0$) and, using (NR-S$_\theta$), that $\pi^S > \underline{\pi}^S$ (i.e., $\lambda_7^{\theta-} = 0$). Equation (15.11), the first-order condition of Problem (P1) with respect to continuation profits, can then be written as $-\left(\lambda_1 + \lambda_2 + \lambda_4^\theta\right) \in \partial Q\left[\pi^S(\theta')\right]$ which, by the non-negativity of the multipliers and the Envelope condition (15.12), implies that there are values $\varepsilon_1 \in \partial Q\left(\pi^S\right)$ and $\varepsilon_2 \in \partial Q\left[\pi^S(\theta')\right]$ such that $\varepsilon_1 \geq \varepsilon_2$. But since $\pi^S(\theta) < \pi^S < \widehat{\pi}^S$ and, by Lemma 8.8, Q is strictly concave, for any $x \in \partial Q\left[\pi^S(\theta')\right]$ and any $y \in \partial Q\left(\pi^S\right)$ it must be that $x > y$. Hence a contradiction. To prove that the inequality is strict for at least one demand realization, observe that if $\pi^S(\theta) = \pi^S$ for all θ, then Lemma 8.8 and Proposition 8.9 imply $\mathbb{E}\left[T(\theta)\right] = 0$. The supplier's profits are then given by $-(1-\delta)\phi ck + \delta\pi^S < \pi^S$, violating the promise-keeping condition. ■

Proof of Theorem 8.11. Once $\pi_t^S \geq \widehat{\pi}^S$, Lemma 8.2 ensures that $\pi_{t+j}^S \geq \widehat{\pi}^S$ for all $j \in N_+$. It only remains to prove that for any efficient relational contract there is a $T < \infty$ such that $\pi_T^S \geq \widehat{\pi}^S$. Let $\pi_t^S\left(h^t\right)$ denote the supplier's expected continuation profits given any optimal relational contract ψ, any period t, and any history $h^t \in H^t$, and let χ denote the probability of the least likely demand realization, i.e., $\chi = \min_{\theta \in \Theta} f_\theta$. The result is trivially true if $\pi_t^S\left(h^t\right) \geq \widehat{\pi}^S$ so suppose $\pi_t^S\left(h^t\right) < \widehat{\pi}^S$ instead. Note that if there is a demand realization such that $\pi_{t+1}^S\left(h^t, \theta_t\right) \geq \widehat{\pi}^S$, then $\Pr\left[\pi_{t+1}^S \geq \widehat{\pi}^S \mid \pi_t^S\left(h^t\right)\right] \geq \chi$. Alternatively, when $\pi_{t+1}^S\left(h^t, \theta_t\right) < \widehat{\pi}^S$ for all θ, Proposition 8.9 requires the supplier's current rents to be zero. Constraint (P-K) then implies

$$-\left(1-\delta\right)\left(1-\phi\right) ck_t\left(h^t\right) + \delta \mathbb{E}\left[\pi_{t+1}^S\left(h^t, \theta_t\right)\right] \geq \pi_t^S\left(h^t\right),$$

which in turn guarantees that there is a demand realization for which

$$\begin{aligned}
\pi_{t+1}^S\left(h^t, \theta\right) &\geq \left[\pi_t^S\left(h^t\right) + \left(1-\delta\right)\left(1-\phi\right) ck_t\left(h^t\right)\right]/\delta \\
&\geq \pi_t^S\left(h^t\right)/\delta \\
&> \pi_t^S\left(h^t\right),
\end{aligned}$$

where the last inequality follows since $\pi_t^S\left(h^t\right) \geq \underline{\pi}^S > 0$ by Proposition 5.1. Furthermore, since the previous lower bound on future continuation profits is strictly increasing in $\pi_t^S\left(h^t\right)$ and $\pi_{\max}^S < \infty$, there must be a finite sequence of demand realizations, say of length $n\left(\psi, t, h^t\right)$, after which $\widehat{\pi}^S$ is reached. Additionally, the recursive structure of the problem ensures that $n\left(\psi, t, h^t\right)$ only depends on the current value of $\pi_t^S\left(h^t\right)$. The fact that the set of equilibrium continuation values is closed and compact then guarantees that there is a finite number $N \in N$ such that $N > n\left(\psi, t, h^t\right)$ for all ψ, t, and h^t. Therefore, for any optimal relational contract ψ, any period t, and any history $h^t \in H^t$, there exists a sequence of demand realizations of length N after which $\pi_{t+N}^S \geq \widehat{\pi}^S$.

Now let A_n denote the set of paths $(\pi_1^S, \pi_2^S, ..., \pi_n^S)$ with $\pi_n^S \geq \widehat{\pi}^S$, and let B_n denote the remaining paths, namely, those paths of length n for which $\pi_n^S < \widehat{\pi}^S$. Observe that for any $(\pi_1^S, \pi_2^S, ..., \pi_n^S)$ in B_n, Lemma 8.2 guarantees that each π_i^S, $i \in \{1, .., n\}$, is strictly less than $\widehat{\pi}^S$. Clearly,

$$\Pr[B_N] = 1 - \Pr[A_N] \leq 1 - \chi^N.$$

In addition,

$$\Pr[B_{2N}] = \Pr[B_N|B_N]\Pr[B_N] \leq \left(1-\chi^N\right)^2,$$

or, more generally,

$$\Pr[B_{M \cdot N}] \leq \left(1-\chi^N\right)^M, \ M = 1, 2, ..., \infty.$$

Therefore,

$$\lim_{M \to \infty} \Pr[B_{M \cdot N}] = 0,$$

which proves the result. ■

Proof of Theorem 8.12. *Part (i).* First note that since the objective function of Problem (P1) is continuous and the correspondence describing the set of contracts satisfying all the constraints as a function of π^S is compact-valued and continuous, the Theorem of the Maximum ensures that the set of maximizer

$$M\left(\pi^S\right) := \left\{\left(k, l, T\left(\theta\right), \pi^S\left(\theta\right)\right) \text{ which solve Problem (P1) given } \pi^S\right\}$$

is non-empty, compact-valued, and upper hemi-continuous in π^S; see Stokey, Lucas, and Prescott (1989). The argument used in the proof of Lemma 15.3 then implies that the value of k is unique across all element of $M\left(\pi^S\right)$. Continuity follows from the upper hemi-continuity of $M\left(\pi^S\right)$.

Part (ii). The proof consists of two parts.

Step 1: I first argue that provided investment is inefficient, $|k\left(\pi^S\right) - k^{FB}|$ is strictly decreasing in π^S. There are two relevant cases depending on whether investment is below or above its efficient level.

Case 1: The proof proceeds by contradiction. Suppose that there is an interval $\left[\pi^{S'}, \pi^{S''}\right] \subseteq \left[\pi^S_{\min}, \widehat{\pi}^S\right]$ in which k is weakly decreasing in π^S and $k(\pi^{S'}) < k^{FB}$. By Lemmas 8.1 and 8.8, $\pi^S + Q\left(\pi^S\right)$ is strictly increasing in π^S whenever $\pi^S < \widehat{\pi}^S$, Proposition 8.10 then ensures that $\pi^{S''} < \widehat{\pi}^S$ and that there is another interval, say $\left[\pi^S_*, \pi^S_{**}\right]$ with $\pi^S_* \neq \pi^S_{**}$, $\pi^S_* \geq \pi^{S''}$ and $\pi^S_{**} \leq \widehat{\pi}^S$, in which k is strictly increasing in π^S. Thus, by the continuity of k, there is a $\pi^S \in (\pi^{S'}, \pi^S_{**})$, call it $\widetilde{\pi}^S$, such that for any sufficiently small $\varepsilon > 0$, $k\left(\widetilde{\pi}^S\right) = k\left(\widetilde{\pi}^S + \varepsilon\right)$. Moreover, (NR-S-K) must be slack at both $\widetilde{\pi}^S$ and $\widetilde{\pi}^S + \varepsilon$. To see why, note that combining constraints (NR-S-K)$(\pi^{S'})$ and (NR-S-K)$(\widetilde{\pi}^S)$ yields

$$\widetilde{\pi}^S > \pi^{S'} \geq (1-\delta)\,\pi^S_{spot}\left(l'\right) + \delta\underline{\pi}^S \geq (1-\delta)\,\pi^S_{spot}\left(l''\right) + \delta\underline{\pi}^S,$$

where the last strict inequality holds since $k(\pi^{S'}) \geq k\left(\widetilde{\pi}^S\right)$ and, by Lemma 8.8, (F-C) is binding at both $\pi^{S'}$ and $\widetilde{\pi}^S$. Hence (NR-S-K) is slack at $\widetilde{\pi}^S$. That (NR-S-K) is also slack at $\widetilde{\pi}^S + \varepsilon$ then follows since $l\left(\widetilde{\pi}^S\right) = l\left(\widetilde{\pi}^S + \varepsilon\right)$ (which holds because $k\left(\widetilde{\pi}^S\right) = k\left(\widetilde{\pi}^S + \varepsilon\right)$ and (F-C) is binding at both $\widetilde{\pi}^S$ and $\widetilde{\pi}^S + \varepsilon$).

Let $\Theta^1\left(\pi^S\right), \Theta^2\left(\pi^S\right) \subseteq \Theta$ denote the set of states in which (NR-S$_\theta$) is binding and slack, respectively, when the supplier's continuation profits are π^S. By the continuity of the multipliers, $\Theta^i\left(\widetilde{\pi}^S\right) = \Theta^i\left(\widetilde{\pi}^S + \varepsilon\right) := \Theta^i$, $i = \{1, 2\}$, for small enough ε. Applying steps 1 and 2 from part (ii) of the proof of Lemma 8.1, the promise-keeping conditions associated with $\widetilde{\pi}^S$ and $\widetilde{\pi}^S + \varepsilon$ can then be written as

$$- (1-\delta)(1-\phi)ck + \sum_{\Theta^1} f_\theta \left[(1-\delta)\beta^s \mathbb{E}[R(k,\theta)] + \delta \underline{\pi}^S\right] + \delta \sum_{\Theta^2} f_\theta \widetilde{\pi}^S = \widetilde{\pi}^S,$$
$$- (1-\delta)(1-\phi)ck + \sum_{\Theta^1} f_\theta \left[(1-\delta)\beta^s \mathbb{E}[R(k,\theta)] + \delta \underline{\pi}^S\right] + \delta \sum_{\Theta^2} f_\theta \left(\widetilde{\pi}^S + \varepsilon\right) = \widetilde{\pi}^S + \varepsilon,$$

where I have used the fact that $\pi^S(\theta) = \pi^S$ whenever both (NR-S-K) and (NR-S$_\theta$) hold with strict inequality. But the last two equations require $\delta \sum_{\Theta^2} f_\theta = 1$, a contradiction since $\delta < 1$ and $\sum_{\Theta^2} f_\theta \leq 1$.

Case 2: Suppose that there is an interval $\left[\pi^{S'}, \pi^{S''}\right] \subseteq \left[\pi^S_{\min}, \widehat{\pi}^S\right]$ in which k is weakly increasing in π^S and $k\left(\pi^{S'}\right) > k^{FB}$. Applying the same arguments as in case 1 guarantees that there is a $\widetilde{\pi}^S < \widehat{\pi}^S$ such that for any sufficiently small $\varepsilon > 0$, $k\left(\widetilde{\pi}^S\right) = k\left(\widetilde{\pi}^S + \varepsilon\right)$. Moreover, by the same logic as in case 1, (NR-S-K) must be slack at $\widetilde{\pi}^S + \varepsilon$. Just as before, the promise-keeping conditions associated with $\widetilde{\pi}^S$ and $\widetilde{\pi}^S + \varepsilon$ can be written as

$$- (1-\delta)(1-\phi)ck + \sum_{\Theta^1} f_\theta \left[(1-\delta)\beta^s \mathbb{E}[R(k,\theta)] + \delta \underline{\pi}^S\right] + \delta \sum_{\Theta^2} f_\theta \pi^S\left(\theta; \widetilde{\pi}^S\right) = \widetilde{\pi}^S,$$
$$- (1-\delta)(1-\phi)ck + \sum_{\Theta^1} f_\theta \left[(1-\delta)\beta^s \mathbb{E}[R(k,\theta)] + \delta \underline{\pi}^S\right] + \delta \sum_{\Theta^2} f_\theta \left(\widetilde{\pi}^S + \varepsilon\right) = \widetilde{\pi}^S + \varepsilon,$$

respectively. Combining the last two equations then yields

$$\delta \sum_{\Theta^2} f_\theta \left[\widetilde{\pi}^S + \varepsilon - \pi^S\left(\theta; \widetilde{\pi}^S\right)\right] = \varepsilon. \tag{15.23}$$

But, by Proposition 8.10 , $\pi^S\left(\theta; \widetilde{\pi}^S\right) \geq \widetilde{\pi}^S$ so that

$$\delta \sum_{\Theta^2} f_\theta \left[\widetilde{\pi}^S + \varepsilon - \pi^S\left(\theta; \widetilde{\pi}^S\right)\right] \leq \delta\varepsilon \sum_{\Theta^2} f_\theta < \varepsilon,$$

where the last inequality holds since $\delta < 1$ and $\sum_{\Theta^2} f_\theta \leq 1$, contradicting (15.23).

Step 2: Finally, I argue that investment is inefficient for any $\pi^S \in [\pi^S_{\min}, \widehat{\pi}^S)$. Suppose to the contrary that $k\left(\pi^S\right) = k^{FB}$ for some $\pi^{S'} < \widehat{\pi}^S$. By step 1, the continuity of k, and the fact that investment is constant for all $\pi^S \geq \widehat{\pi}^S$, it follows that $k\left(\pi^S\right) = k^{FB}$ for all $\pi^S \in \left[\pi^{S'}, \pi^S_{\max}\right]$. Proposition 8.10 then implies that $\pi^S + Q\left(\pi^S\right) = -ck^{FB} + \mathbb{E}\left[R\left(k^{FB}, \theta\right)\right]$, which in turn ensures that $Q'_-\left(\pi^S\right) = 1$ for all $\pi^S \in (\pi^{S'}, \pi^S_{\max}]$. But then, by the definition of $\widehat{\pi}^S$, $\widehat{\pi}^S \leq \pi^{S'}$—a contradiction.

Part (iii). That $k\left(\pi^S\right) \neq k^{FB}$ was proved in step 2 of part (ii). To prove the rest of the claim, recall from Proposition 8.3 that $k\left(\widehat{\pi}^S\right)$ can be above or below its efficient level. The result then follows from part (i): the single-valuedness and continuity of k. ■

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